A FIRST COURSE IN DYNAMICS

with a Panorama of Recent Developments

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Introduction

This chapter is a prelude to this book. It first describes in general terms what the discipline of dynamical systems is about. The following sections contain a large number of examples. Some of the problems treated later in the book appear here for the first time.

1.1 DYNAMICS

What is a dynamical system? It is dynamical, something happens, something changes over time. How do things change in nature? Galileo Galilei and Isaac Newton were key players in a revolution whose central tenet is *Nature obeys unchanging laws that mathematics can describe.* Things behave and evolve in a way determined by fixed rules. The prehistory of dynamics as we know it is the development of the laws of mechanics, the pursuit of exact science, and the full development of classical and celestial mechanics. The Newtonian revolution lies in the fact that the principles of nature can be expressed in terms of mathematics, and physical events can be predicted and designed with mathematical certainty. After mechanics, electricity, magnetism, and thermodynamics, other natural sciences followed suit, and in the social sciences quantitative deterministic descriptions also have taken a hold.

1.1.1 Determinism Versus Predictability

The key word is determinism: Nature obeys unchanging laws. The regularity of celestial motions has been the primary example of order in nature forever:

> God said, let there be lights in the firmament of the heavens to divide the day from the night and let them be for signs and for seasons and for days and years.

The successes of classical and especially celestial mechanics in the eighteenth and nineteenth centuries were seemingly unlimited, and Pierre Simon de Laplace felt justified in saying (in the opening passage he added to his 1812 *Philosophical Essay*

on Probabilities):

> We ought then to consider the present state of the universe as the effects of its previous state and as the cause of that which is to follow. An intelligence that, at a given instant, could comprehend all the forces by which nature is animated and the respective situation of the beings that make it up, if moreover it were vast enough to submit these data to analysis, would encompass in the same formula the movements of the greatest bodies of the universe and those of the lightest atoms. For such an intelligence nothing would be uncertain, and the future, like the past, would be open to its eyes.[1]

The enthusiasm in this 1812 overture is understandable, and this forceful description of determinism is a good anchor for an understanding of one of the basic aspects of dynamical systems. Moreover, the titanic life's work of Laplace in celestial mechanics earned him the right to make such bold pronouncements. There are some problems with this statement, however, and a central mission of dynamical systems and of this book is to explore the relation between determinism and predictability, which Laplace's statement misses. The history of the modern theory of dynamical systems begins with Henri Jules Poincaré in the late nineteenth century. Almost 100 years after Laplace he wrote a summary rejoinder:

> If we could know exactly the laws of nature and the situation of the universe at the initial instant, we should be able to predict exactly the situation of this same universe at a subsequent instant. But even when the natural laws should have no further secret for us, we could know the initial situation only *approximately*. If that permits us to foresee the subsequent situation *with the same degree of approximation*, this is all we require, we say the phenomenon has been predicted, that it is ruled by laws. But this is not always the case; it may happen that slight differences in the initial conditions produce very great differences in the final phenomena; a slight error in the former would make an enormous error in the latter. Prediction becomes impossible and we have the fortuitous phenomenon.[2]

His insights led to the point of view that underlies the study of dynamics as it is practiced now and as we present it in this book: The study of long-term asymptotic behavior, and especially that of its qualitative aspects, requires direct methods that do not rely on prior explicit calculation of solutions. And in addition to the qualitative (geometric) study of a dynamical system, probabilistic phenomena play a role.

A major motivation for the study of dynamical systems is their pervasive importance in dealing with the world around us. Many systems evolve continuously in time, such as those in mechanics, but there are also systems that naturally evolve in discrete steps. We presently describe models of, for example, butterfly populations, that are clocked by natural cycles. Butterflies live in the summer, and

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[1] Pierre Simon marquis de Laplace, *Philosophical Essay on Probabilities*, translated from the fifth French edition of 1925 by Andrew I. Dale, Springer-Verlag, New York, 1995, p. 2.

[2] Henri Jules Poincaré, Science et méthode, Section IV.II., Flammarion 1908; see *The Foundations of Science; Science and Hypothesis, The Value of science, Science and Method*, translated by George Bruce Halsted, The Science Press, Lancaster, PA, 1946, pp. 397f; *The Value of Science: Essential Writings of Henri Poincaré*, edited by Stephen Jay Gould, Modern Library, 2001.

we discuss laws describing how next summer's population size is determined by that of this summer. There are also ways of studying a continuous-time system by making it look like a discrete-time system. For example, one might check on the moon's position precisely every 24 hours. Or one could keep track of where it rises any given day. Therefore we allow dynamical systems to evolve in discrete steps, where the same rule is applied repeatedly to the result of the previous step.

This is important for another reason. Such stepwise processes do not only occur in the world around us, but also in our minds. This happens whenever we go through repeated steps on our way to the elusive perfect solution. Applied to such procedures, dynamics provides insights and methods that are useful in analysis. We show in this book that important facts in analysis are consequences of dynamical facts, even of some rather simple ones: The Contraction Principle (Proposition 2.2.8, Proposition 2.2.10, Proposition 2.6.10) gives the Inverse-Function Theorem 9.2.2 and the Implicit-Function Theorem 9.2.3. The power of dynamics in situations of this kind has to do with the fact that various problems can be approached with an iterative procedure of successive approximation by improved guesses at an answer. Dynamics naturally provides the means to understand where such a procedure leads.

1.1.2 Dynamics in Analysis

Whenever you use a systematic procedure to improve a guess at a solution you are likely to have found a way of using dynamics to solve your problem exactly. To begin to appreciate the power of this approach it is important to understand that the iterative processes dynamics can handle are not at all required to operate on numbers only. They may manipulate quite complex classes of objects: numbers, points in Euclidean space, curves, functions, sequences, mappings, and so on. The possibilities are endless, and dynamics can handle them all. We use iteration schemes on functions in Section 9.4, mappings in Section 9.2.1 and sequences in Section 9.5. The beauty of these applications lies in the elegance, power, and simplicity of the solutions and insights they provide.

1.1.3 Dynamics in Mathematics

The preceding list touches only on a portion of the utility of dynamical systems in understanding mathematical structures. There are others, where insights into certain patterns in some branches of mathematics are most easily obtained by perceiving that underlying the structure in question is something of a dynamical nature that can readily be analyzed or, sometimes, has been analyzed already. This is a range of applications of dynamical ideas that is exciting because it often involves phenomena of a rich subtlety and variety. Here the beauty of applying dynamical systems lies in the variety of behaviors, the surprising discovery of order in bewildering complexity, and in the coherence between different areas of mathematics that one may discover. A little later in this introductory chapter we give some simple examples of such situations.

■ **EXERCISES**

In these exercises you are asked to use a calculator to play with some simple iterative procedures. These are not random samples, and we return to several of these in due course. In each exercise you are given a function f as well as a number x_0. The assignment is to consider the sequence defined recursively by the given initial value and the rule $x_{n+1} = f(x_n)$. Compute enough terms to describe what happens in the long run. If the sequence converges, note the limit and endeavor to determine a closed expression for it. Note the number of steps you needed to compute to see the pattern or to get a good approximation of the limit.

■ **Exercise 1.1.1** $f(x) = \sqrt{2 + x}$, $x_0 = 1$.

■ **Exercise 1.1.2** $f(x) = \sin x$, $x_0 = 1$. Use the *degree* setting on your calculator – this means that (in radians) we actually compute $f(x) = \sin(\pi x/180)$.

■ **Exercise 1.1.3** $f(x) = \sin x$, $x_0 = 1$. Use the *radian* setting here and forever after.

■ **Exercise 1.1.4** $f(x) = \cos x$, $x_0 = 1$.

■ **Exercise 1.1.5**
$$f(x) = \frac{x \sin x + \cos x}{1 + \sin x}, \quad x_0 = 3/4.$$

■ **Exercise 1.1.6** $f(x) = \{10x\} = 10x - \lfloor 10x \rfloor$ (fractional part), $x_0 = \sqrt{1/2}$.

■ **Exercise 1.1.7** $f(x) = \{2x\}$, $x_0 = \sqrt{1/2}$.

■ **Exercise 1.1.8**
$$f(x) = \frac{5 + x^2}{2x}, \quad x_0 = 2.$$

■ **Exercise 1.1.9** $f(x) = x - \tan x$, $x_0 = 1$.

■ **Exercise 1.1.10** $f(x) = kx(1 - x)$, $x_0 = 1/2$, $k = 1/2, 1, 2, 3.1, 3.5, 3.83, 3.99, 4$.

■ **Exercise 1.1.11** $f(x) = x + e^{-x}$, $x_0 = 1$.

1.2 DYNAMICS IN NATURE

1.2.1 Antipodal Rabbits

Rabbits are not indigenous to Australia, but 24 wild European rabbits were introduced by one Thomas Austin near Geelong in Southern Victoria around 1860, with unfortunate consequences. Within a decade they were rampant across Victoria, and within 20 years millions had devastated the land, and a prize of £25,000 was advertized for a solution. By 1910 their descendants had spread across most of the continent. The ecological impact is deep and widespread and has been called a national tragedy. The annual cost to agriculture is estimated at AU$600 million. The unchecked growth of their population makes an interesting example of a dynamical system.

In modeling the development of this population we make a few choices. Its large size suggests to count it in millions, and when the number of rabbits is

expressed as x million then x is not necessarily an integer. After all, the initial value is 0.000024 million rabbits. Therefore we measure the population by a real number x. As for time, in a mild climate rabbits – famously – reproduce continuously. (This is different for butterflies, say, whose existence and reproduction are strictly seasonal; see Section 1.2.9.) Therefore we are best served by taking the time variable to be a real number as well, t, say. Thus we are looking for ways of describing the number of rabbits as a function $x(t)$ of time.

To understand the dependence on time we look at what rabbits do: They eat and reproduce. Australia is large, so they can eat all they want, and during any given time period Δt a fixed percentage of the (female) population will give birth and a (smaller) percentage will die of old age (there are no natural enemies). Therefore the increment $x(t + \Delta t) - x(t)$ is proportional to $x(t)\Delta t$ (via the difference of birth and death rates). Taking a limit as $\Delta t \to 0$ we find that

$$(1.2.1) \qquad \frac{dx}{dt} = kx,$$

where k represents the (fixed) relative growth rate of the population. Alternatively, we sometimes write $\dot{x} = kx$, where the dot denotes differentiation with respect to t. By now you should recognize this model from your calculus class.

It is the unchanging environment (and biology) that gives rise to this unchanging evolution law and makes this a dynamical system of the kind we study. The differential equation (1.2.1), which relates x and its rate of change, is easy to solve: Separate variables (all x on the left, all t on the right) to get $(1/x)dx = k\,dt$ and integrate this with respect to t using substitution:

$$\log |x| = \int \frac{1}{x}\,dx = \int k\,dt = kt + C,$$

where log is the *natural* logarithm. Therefore, $|x(t)| = e^C e^{kt}$ with $e^C = |x(0)|$ and we find that

$$(1.2.2) \qquad x(t) = x(0)e^{kt}.$$

■ **Exercise 1.2.1** Justify the disappearance of the absolute value signs above.

■ **Exercise 1.2.2** If $x(0) = 3$ and $x(4) = 6$, find $x(2)$, $x(6)$, and $x(8)$.

1.2.2 The Leaning Rabbits of Pisa

In the year 1202, Leonardo of Pisa considered a more moderate question regarding rabbits, which we explore in Example 2.2.9 and Section 3.1.9. The main differences to the large-scale Australian model above are that the size of his urban yard limited him to small numbers of rabbits and that with such a small number the population growth does not happen continuously, but in relatively substantial discrete steps. Here is the problem as he posed it:[3]

How many pairs of rabbits can be bred from one pair in one year?

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[3] Leonardo of Pisa: Liber abaci (1202), published in *Scritti di Leonardo Pisano*, Rome, B. Boncompagni, 1857; see p. 3 of Dirk J. Struik, *A Source Book in Mathematics 1200–1800*, Princeton, NJ, Princeton University Press, 1986.

A man has one pair of rabbits at a certain place entirely surrounded by a wall. We wish to know how many pairs can be bred from it in one year, if the nature of these rabbits is such that they breed every month one other pair and begin to breed in the second month after their birth. Let the first pair breed a pair in the first month, then duplicate it and there will be 2 pairs in a month. From these pairs one, namely the first, breeds a pair in the second month, and thus there are 3 pairs in the second month. From these in one month two will become pregnant, so that in the third month 2 pairs of rabbits will be born. Thus there are 5 pairs in this month. From these in the same month 3 will be pregnant, so that in the fourth month there will be 8 pairs ... [We have done this] by combining the first number with the second, hence 1 and 2, and the second with the third, and the third with the fourth ...

In other words, he came up with a sequence of numbers (of pairs of rabbits) governed by the recursion $b_{n+1} = b_n + b_{n-1}$ and chose starting values $b_0 = b_1 = 1$, so the sequence goes 1, 1, 2, 3, 5, 8, 13, ... Does this look familiar? (Hint: As the son of Bonaccio, Leonardo of Pisa was known as filius Bonacci or "son of good nature"; Fibonacci for short.) Here is a question that can be answered easily with a little bit of dynamics: How does his model compare with the continuous exponential-growth model above?

According to exponential growth one should expect that once the terms get large we always have $b_{n+1} \approx ab_n$ for some constant a independent of n. If we pretend that we have actual equality, then the recursion formula gives

$$a^2 b_n = ab_{n+1} = b_{n+2} = b_{n+1} + b_n = (a+1)b_n,$$

so we must have $a^2 = a + 1$. The quadratic formula then gives us the value of the growth constant a.

■ **Exercise 1.2.3** Calculate a.

Note, however, that we have only shown that *if* the growth is eventually exponential, then the growth constant is this a, not that the growth is eventually exponential. (If we *assume* the recursion $b_{n+1} = 1$ leads to exponential growth, we could come up with a growth parameter if we are quick enough to do it before getting a contradiction.) Dynamics provides us with tools that enable us to verify this property easily in various different ways (Example 2.2.9 and Section 3.1.9). In Proposition 3.1.11 we even convert this recursively defined sequence into closed form.

The value of this asymptotic ratio was known to Johannes Kepler. It is the golden mean or the divine proportion. In his 1619 book *Harmonices Mundi* he wrote (on page 273):

there is the ratio which is never fully expressed in numbers and cannot be demonstrated by numbers in any other way, except by a long series of numbers gradually approaching it: this ratio is called *divine*, when it is perfect, and it rules in various ways throughout the dodecahedral wedding. Accordingly, the following consonances begin to shadow forth that ratio: 1:2 and 2:3 and 3:5 and 5:8. For it exists most imperfectly in 1:2, more perfectly in 5:8, and still more perfectly if we add 5 and 8 to make 13 and take 8 as the numerator . . . .[4]

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[4] Johannes Kepler, *Epitome of Copernican Astronomy & Harmonies of the World*, Amherst, NY, Prometheus Books, 1995.

We note in Example 15.2.5 that these Fibonacci ratios are the optimal rational approximations of the golden mean.

■ **Exercise 1.2.4** Express $1 + 1 + 2 + 3 + \cdots + b_n$ in terms of b_{n+2}.

1.2.3 Fine Dining

Once upon a time lobsters were so abundant in New England waters that they were poor man's food. It even happened that prisoners in Maine rioted to demand to be fed something other than lobsters for a change. Nowadays the haul is less abundant and lobsters have become associated with fine dining. One (optimistic?) model for the declining yields stipulates that the catch in any given year should turn out to be the average of the previous two years' catches.

Using again a_n for the number of lobsters caught in the year n, we can express this model by a simple recursion relation:

(1.2.3)
$$a_{n+1} = a_{n-1}/2 + a_n/2.$$

As initial values one can take the Maine harvests of 1996 and 1997, which were 16,435 and 20,871 (metric) tons, respectively. This recursion is similar to the one for the Fibonacci numbers, but in this case no exponential growth is to be expected. One can see from the recursion that all future yields should be between the two initial data. Indeed, 1997 was a record year. In Proposition 3.1.13 we find a way of giving explicit formulas for future yields, that is, we give the yield in an arbitrary year n in a closed form as a function of n.

This situation as well as the Fibonacci rabbit problem are examples where time is measured in discrete steps. There are many other examples where this is natural. Such a scenario from population biology is discussed in Section 1.2.9. Other biological examples arise in genetics (gene frequency) or epidemiology. Social scientists use discrete-time models as well (commodity prices, rate of spread of a rumor, theories of learning that model the amount of information retained for a given time).

1.2.4 Turning Over a New Leaf

The word phyllotaxis comes from the words phyllo=leaf and taxis=order or arrangement. It refers to the way leaves are arranged on twigs, or other plant components on the next larger one. The seeds of a sunflower and of a pine cone are further examples. A beautiful description is given by Harold Scott Macdonald Coxeter in his *Introduction to Geometry*. That regular patterns often occur is familiar from sunflowers and pineapples.

In some species of trees the leaves on twigs are also arranged in regular patterns. The pattern varies by species. The simplest pattern is that of leaves alternating on opposite sides of the twig. It is called (1, 2)-phyllotaxis: Successive leaves are separated by a half-turn around the twig. The leaves of elms exhibit this pattern, as do hazel leaves.[5] Adjacent leaves may also have a (2/3) turn between them, which would be referred to as (2, 3)-phyllotaxis. Such is the case with beeches. Oak trees

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[5] On which the first author of this book should be an expert!

show a (3, 5)-pattern, poplars a (5, 8), and willows, (8, 13)-phyllotaxis. Of course, the pattern may not always be attained to full precision, and in some plants there are transitions between different patterns as they grow.

The diamond-shaped seeds of a sunflower are packed densely and regularly. One may perceive a spiral pattern in their arrangement, and, in fact, there are always two such patterns in opposite directions. The numbers of spirals in the two patterns are successive Fibonacci numbers. The seeds of a fir cone exhibit spirals as well, but on a cone rather than flat ones. These come in two families, whose numbers are again successive Fibonacci numbers.

Pineapples, too, exhibit spiral patterns, and, because their surface is composed of approximately hexagonal pieces, there are three possible directions in which one can perceive spirals. Accordingly, one may find 5, 8, and 13 spirals: 5 sloping up gently to the right, say, 8 sloping up to the left, and 13 sloping quite steeply right.

The observation and enjoyment of these beautiful patterns is not new. They were noticed systematically in the nineteenth century. But an explanation for why there are such patterns did not emerge particularly soon. In fact, the case is not entirely closed yet.

Here is a model that leads to an explanation of how phyllotaxis occurs. The basic growth process of this type consists of buds (primordia) of leaves or seeds growing out of a center and then moving away from it according to three rules proposed in 1868 by the self-taught botanist Wilhelm Friedrich Benedikt Hofmeister, while he was professor and director of the botanical garden in Heidelberg:

(1) New buds form at regular intervals, far from the old ones.
(2) Buds move radially from the center.
(3) The growth rate decreases as one moves outward.

A physical experiment designed to mimic these three *Hofmeister rules* produces spiral patterns of this Fibonacci type, so from these rules one should be able to infer that spiral patterns must occur. This has been done recently with methods of the kind that this book describes.[6]

Here is a description of how dynamics may help. To implement the Hofmeister rules we model the situation by a family of $N+1$ concentric circles of radius r^k ($k = 0, \ldots, N$), where r stands for growth rate, and we put a bud on each circle. The angle (with respect to the origin) between one bud and the next is θ_k. Possible patterns are now parametrized by angles $(\theta_0, \ldots, \theta_N)$. This means that the "space of plants" is a *torus*; see Section 2.6.4. When a new bud appears on the unit circle, all other buds move outward one circle. The angle of the new bud depends on all previous angles, so we get a map sending old angles θ_k to new angles Θ_k by

$$\Theta_0 = f(\theta_0, \ldots, \theta_N), \quad \Theta_1 = \theta_0, \ldots, \Theta_N = \theta_{N-1}.$$

Now f has to be designed to reflect the first Hofmeister rule. One way to do this is to define a natural potential energy to reflect "repulsion" between buds and choosing

[6] Pau Atela, Christophe Golé, and Scott Hotton: A dynamical system for plant pattern formation: A rigorous analysis, *Journal of Nonlinear Science* **12** (2002), no. 6, pp. 641–676.

$f(\theta_0, \ldots, \theta_N)$ to be the minimum. A natural potential is

$$W(\Theta) = \sum_{k=0}^{N} U(\|r^k e^{i\theta_k} - e^{i\Theta}\|),$$

where $U(x) = 1/x^s$ for some $s > 0$. A simpler potential that gives the same qualitative behavior is $W(\Theta) = \max_{0 \le k \le N} U(\|r^k e^{i\theta_k} - e^{i\Theta}\|)$. With either choice one can show that regular spirals (that is, $\theta_0 = \cdots = \theta_N$) are attracting fixed points (Section 2.2.7) of this map. This means that spirals will appear naturally. A result of the analysis is furthermore that the Fibonacci numbers also must appear.

1.2.5 Variations on Exponential Growth

In the example of a rabbit population of Section 1.2.1 it is natural to expect a positive growth parameter k in the equation $\dot{x} = kx$. This coefficient, however, is the difference between rates of reproduction and death. For the people of some western societies, the reproduction rate has declined so much as to be lower than the death rate. The same model still applies, but with $k < 0$ the solution $x(t) = x(0)e^{kt}$ describes an exponentially shrinking population.

The same differential equation $\dot{x} = kx$ comes up in numerous simple models because it is the simplest differential equation in one variable.

Radioactive decay is a popular example: It is an experimental fact that of a particular radioactive substance a specific percentage will decay in a fixed time period. As before, this gives $\dot{x} = kx$ with $k < 0$. In this setting the constant k is often specified by the *half-life*, which is the time T such that $x(t + T) = x(t)/2$. Depending on the substance, this time period may be minute fractions of a second to thousands of years. This is important in regard to the disposal of radioactive waste, which often has a long half-life, or radioactive contamination. Biology laboratories use radioactive phosphorus as a marker, which has a half-life of a moderate number of days. A spill on the laboratory bench is usually covered with plexiglas for some two weeks, after which the radiation has sufficiently diminished. On the other hand, a positive effect of radioactive decay is the possibility of radioisotope dating, which can be used to assess the age of organic or geologic samples. Unlike in population biology, the exponential decay model of radioactivity needs no refinements to account for real data. It is an exact law of nature.

■ **Exercise 1.2.5** Express the half-life in terms of k, and vice versa.

The importance of the simple differential equation $\dot{x} = kx$ goes far beyond the collection of models in which it appears, however many of these there may be. It also comes up in the study of more complicated differential equations as an approximation that can illuminate some of the behavior in the more complicated setting. This approach of *linearization* is of great importance in dynamical systems.

1.2.6 The Doomsday Model

We now return to the problem of population growth. Actual population data show that the world population has grown with increasing rapidity. Therefore we should consider a modification of the basic model that takes into account the progress of

civilization. Suppose that with the growth of the population the growing number of researchers manages to progressively decrease the death rate and increase fertility as well. Assuming, boldly, that these improvements make the relative rate of increase in population a small positive power x^ϵ of the present size x (rather than being constant k), we find that

$$\frac{dx}{dt} = x^{1+\epsilon}.$$

As before, this is easy to solve by separating variables:

$$t + C = \int x^{-1-\epsilon}\, dx = -x^{-\epsilon}/\epsilon$$

with $C = -x(0)^{-\epsilon}/\epsilon$, so $x(t) = (x(0)^{-\epsilon} - \epsilon t)^{-1/\epsilon}$, which becomes infinite for $t = 1/(\epsilon x(0)^\epsilon)$. Population explosion indeed!

As far as biology is concerned, this suggests refining our model. Clearly, our assumptions on the increasing growth rate were too generous (ultimately, resources are limited). As an example in differential equations this is instructive, however: There are reasonable-looking differential equations that have divergent solutions.

1.2.7 **Predators**

The reason rabbits have not over taken over the European continent is that there have always been predators around to kill rabbits. This has interesting effects on the population dynamics, because the populations of predators and their prey interact: A small number of rabbits decreases the predator population by starvation, which tends to increase the rabbit population. Thus one expects a stable equilibrium – or possibly oscillations.

Many models of interacting populations of predator and prey were proposed independently by Alfred Lotka and Vito Volterra. A simple one is the *Lotka–Volterra equation*:

$$\frac{dx}{dt} = a_1 x + c_1 xy$$
$$\frac{dy}{dt} = a_2 x + c_2 xy,$$

where $a_1, c_2 > 0$ and $a_2, c_1 < 0$, that is, x is the prey population, which would grow on its own ($a_1 > 0$) but is diminished by the predator ($c_1 < 0$), while y is the predator, which would starve if alone ($a_2 < 0$) and grows by feeding on its prey ($c_2 > 0$). Naturally, we take x and y positive. This model assumes that there is no delay between causes and effects due to the time of gestation or egg incubation. This is reasonable when the time scale of interest is not too short. Furthermore, choosing time continuously is most appropriate when generations overlap substantially. Populations with nonoverlapping generations will be treated shortly.

There is an equilibrium of species at $(a_2/c_2, a_1/c_1)$. Any other initial set of populations turns out to result in oscillations of the numbers of predator and prey. To see this, use the chain rule to verify that

$$E(x, y) := x^{-a_2} e^{-c_2 x} y^{a_1} e^{c_1 y}$$

is constant along orbits, that is, $(d/dt)E(x(t), y(t)) = 0$. This means that the

solutions of the Lotka–Volterra equation must lie on the curves $E(x, y) = \text{const}$. These curves are closed.

1.2.8 Horror Vacui

The Lotka–Volterra equation invites a brief digression to a physical system that shows a different kind of oscillatatory behavior. Its nonlinear oscillations have generated much interest, and the system has been important for some developments in dynamics.

The Dutch engineer Balthasar van der Pol at the Science Laboratory of the Philips Light Bulb Factory in Eindhoven modeled a vacuum tube circuit by the differential equation

$$\frac{d^2x}{dt^2} + \epsilon(x^2 - 1)\frac{dx}{dt} + x = 0,$$

which can be rewritten using $y = dx/dt$ as

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = \epsilon(1 - x^2)y - x.$$

If $\epsilon = 1$, the origin is a repeller (Definition 2.3.6). However, solutions do not grow indefinitely, because there is a periodic solution that circles around the origin. Indeed, for $\epsilon = 0$ there are only such solutions, and for $\epsilon = 1$ one of these circles persists in deformed shape, and all other solutions approach it ever more closely as $t \to +\infty$. The numerically computed picture in Figure 1.2.1 shows this clearly. The curve is called a *limit cycle*.

As an aside we mention that there is also the potential for horrifying complexity in a vacuum tube circuit. In 1927, van der Pol and J. van der Mark reported on experiments with a "relaxation oscillator" circuit built from a capacitor and a neon lamp (this is the nonlinear element) and a periodic driving voltage. (A driving voltage corresponds to putting a periodic term on the right-hand side of the van der Pol equation above.) They were interested in the fact that, in contrast to a linear oscillator (such as a violin string), which exhibits multiples of a base frequency, these oscillations were at "submultiples" of the basic frequency, that is, half that frequency, a third, and so on down to 1/40th, as the driving voltage increased. They
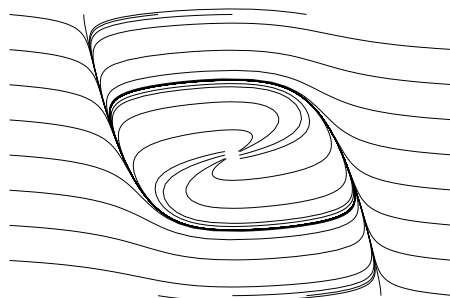

Figure 1.2.1. The van der Pol equation.

obtained these frequencies by listening "with a telephone coupled loosely in some way to the system" and reported that

> Often an irregular noise is heard in the telephone receivers before the frequency jumps to the next lower value. However, this is a subsidiary phenomenon, the main effect being the regular frequency demultiplication.

This irregular noise was one of the first experimental encounters with what was to become known as chaos, but the time was not ripe yet.[7]

1.2.9 **The Other Butterfly Effect**[8]

Population dynamics is naturally done in discrete-time steps when generations do not overlap. This was imposed somewhat artificially in the problem posed by Leonardo of Pisa (Section 1.2.2). For many populations this happens naturally, especially insects in temperate zones, including many crop and orchard pests. A pleasant example is a butterfly colony in an isolated location with a fairly constant seasonal cycle (unchanging rules and no external influence). There is no overlap at all between the current generation (this summer) and the next (next summer). We would like to know how the size of the population varies from summer to summer. There may be plenty of environmental factors that affect the population, but by assuming unchanging rules we ensure that next summer's population depends only on this summer's population, and this dependence is the same every year. That means that the only parameter in this model that varies at all is the population itself. Therefore, up to choosing some fixed constants, the evolution law will specify the population size next summer as a function of this summer's population only. The specific evolution law will result from modeling this situation according to our understanding of the biological processes involved.

1. Exponential growth. For instance, it is plausible that a larger population is likely to lay more eggs and produce a yet larger population next year, proportional, in fact, to the present population. Denoting the present population by x, we then find that next year's population is $f(x) = kx$ for some positive constant k, which is the average number of offspring per butterfly. If we denote the population in year i by x_i, we therefore find that $x_{i+1} = f(x_i) = kx_i$ and in particular that $x_1 = kx_0, x_2 = kx_1 = k^2 x_0$, and so on, that is, $x_i = k^i x_0$; the population grows exponentially. This looks much like the exponential–growth problem as we analyzed it in continuous time.

2. Competition. A problem familiar from public debate is sustainability, and the exponential growth model leads to large populations relatively rapidly. It is more realistic to take into account that a large population will run into problems with limited food supplies. This will, by way of malnutrition or starvation, reduce the

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[7] B. van der Pol, J. van der Mark, Frequency demultiplication, *Nature* **120** (1927), 363–364.

[8] This is a reference to the statement of Edward Lorenz (see Section 13.3) that a butterfly may flutter by in Rio and thereby cause a typhoon in Tokyo a week later. Or maybe to butterfly ballots in the 2000 Florida election?

number of butterflies available for egg-laying when the time comes. A relatively small number of butterflies next year is the result.

The simplest rule that incorporates such more sensible qualitative properties is given by the formula $f(x) = k(1 - \alpha x)x$, where x is the present number of butterflies. This rule is the simplest because we have only adduced a linear correction to the growth rate k. In this correction α represents the rate at which fertility is reduced through competition. Alternatively, one can say that $1/\alpha$ is the maximal possible number of butterflies; that is, if there are $1/\alpha$ butterflies this year, then they will eat up all available food before getting a chance to lay their eggs; hence they will starve and there will be no butterflies next year. Thus, if again x_i denotes the butterfly population in the year i, starting with $i = 0$, then the evolution is given by $x_{i+1} = kx_i(1 - \alpha x_i) =: f(x_i)$. This is a deterministic mathematical model in which every future state (size of the butterfly colony) can be computed from this year's state. One drawback is that populations larger than $1/\alpha$ appear to give negative populations the next year, which could be avoided with a model such as $x_{i+1} = x_i e^{k(1 - x_i)}$. But tractability makes the simpler model more popular, and it played a significant role in disseminating to scientists the important insight that simple models can have complicated long-term behaviors.[9]

One feature reminiscent of the exponential-growth model is that, for populations much smaller than the limit population, growth is indeed essentially exponential: If $\alpha x \ll 1$, then $1 - \alpha x \approx 1$ and thus $x_{i+1} \approx kx_i$; hence $x_n \approx k^n x_0$ – but only so long as the population stays small. This makes intuitive sense: The population is too small to suffer from competition for food, as a large population would.

Note that we made a slip in the previous paragraph: The sequence $x_n \approx k^n x_0$ grows exponentially *if $k > 1$*. If this is not the case, then the butterfly colony becomes extinct. An interesting interplay between reproduction rates and the carrying capacity influences the possibilities here.

3. Change of variable. To simplify the analysis of this system it is convenient to make a simple change of variable that eliminates the parameter α. We describe it with some care here, because changing variables is an important tool in dynamics.

Write the evolution law as $x' = kx(1 - \alpha x)$, where x is the population in one year and x' the population in the next year. If we rescale our units by writing $y = \alpha x$, then we must set

$$y' = \alpha x' = \alpha kx(1 - \alpha x) = ky(1 - y).$$

In other words, we now iterate the map $g(y) = ky(1 - y)$. The relationship between the maps f and g is given by $g(y) = h^{-1}(f(h(y)))$, where $h(y) = y/\alpha = x$. This can be read as "go from new variable to old, apply the old map, and then go to the new variable again."

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[9] As its title shows, getting this message across was the aim of an influential article by Robert M. May, Simple Mathematical Models with Very Complicated Dynamics, *Nature* **261** (1976), 459–467. This article also established the quadratic model as the one to be studied. A good impression of the effects on various branches of biology is given by James Gleick, *Chaos, Making a New Science*, Viking Press, New York, 1987, pp. 78ff.

The effect of this change of variable is to normalize the competition factor α to 1. Since we never chose specific units to begin with, let's rename the variables and maps back to x and f.

4. The logistic equation. We have arrived at a model of this system that is represented by iterations of

$$f(x) = kx(1 - x).$$

This map f is called the *logistic map* (or logistic family, because there is a parameter), and the equation $x' = kx(1 - x)$ is called the logistic equation. The term logistic comes from the French *logistique*, which in turn derived from *logement*, the lodgment of soldiers. We also refer to this family of maps as the *quadratic family*. It was introduced in 1845 by the Belgian sociologist and mathematician Verhulst.[10]

From the brief discussion before the preceding subsection it appears that the case $k \leq 1$ results in inevitable extinction. This is indeed the case. For $k < 1$, this is clear because $kx(1 - x) < kx$, and for $k = 1$ it is not hard to verify either, although the population decay is not exponential in this case. By contrast, large values of k should be good for achieving a large population. Or maybe not. The problem is that too large a population will be succeeded by a less numerous generation. One would hope that the population settles to an agreeable size in due time, at which there is a balance between fertility and competition.

■ **Exercise 1.2.6** Prove that the case $k = 1$ results in extinction.

Note that, unlike in the simpler exponential growth model, we now refrained from writing down an explicit formula for x_n in terms of x_0. This formula is given by polynomials of order 2^n. Even if one were to manage to write them down for a reasonable n, the formulas would not be informative. We will, in due course, be able to say quite a bit about the behavior of this model. At the moment it makes sense to explore it a little to see what kind of behavior occurs. Whether the initial size of the population matters, we have not seen yet. But changing the parameter k certainly is likely to make a difference, or so one would hope, because it would be a sad model indeed that predicts certain extinction all the time. The reasonable range for k is from 0 to 4. [For $k > 4$, it predicts that a population size of 1/2 is followed two years later by a negative population, which makes little biological sense. This suggests that a slightly more sophisticated (nonlinear) correction rule would be a good idea.]

5. Experiments. Increasing k should produce the possibility of a stable population, that is, to allow the species to avoid extinction. So let's start working out the model for some $k > 1$. A simpleminded choice would be $k = 2$, halfway between 0 and 4.

■ **Exercise 1.2.7** Starting with $x = 0.01$, iterate $2x(1 - x)$ until you discern a clear pattern.

[10] Pierre-François Verhulst, Récherches mathématiques sur la loi d'accroissement de la population, *Nouvelles Mémoires de l'Academie Royale des Sciences et Belles-Lettres de Bruxelles* **18** (1845), 1–38.

Starting from a small population, one obtains steady growth and eventually the population levels off at 1/2. This is precisely the behavior one should expect from a decent model. Note that steady states satisfy $x = 2x(1 - x)$, of which 0 and 1/2 are the only solutions.

■ **Exercise 1.2.8** Starting with $x = 0.01$ iterate $1.9x(1 - x)$ and $2.1x(1 - x)$ until you discern a clear pattern.

If k is a little less than 2, the phenomenon is rather the same, for k a little bigger it also goes that way, except for slightly overshooting the steady-state population.

■ **Exercise 1.2.9** Starting with $x = 0.01$, iterate $3x(1 - x)$ and $2.9x(1 - x)$ until you discern a clear pattern.

For $k = 3$, the ultimate behavior is about the same, but the way the population settles down is a little different. There are fairly substantial oscillations of too large and too small population that die out slowly, whereas for k near 2 there was only a hint of this behavior, and it died down fast. Nevertheless, an ultimate steady state still prevails.

■ **Exercise 1.2.10** Starting with $x = 0.01$, iterate $3.1x(1 - x)$ until you discern a clear pattern.

For $k = 3.1$, there are oscillations of too large and too small as before. They do get a little smaller, but this time they do not die down all the way. With a simple program one can iterate this for quite a while and see that no steady state is attained.

■ **Exercise 1.2.11** Starting with $x = 0.66$, iterate $3.1x(1 - x)$ until you discern a clear pattern.

In the previous experiment, there is the possibility that the oscillations die down so slowly that the numerics fail to notice. Therefore, as a control, we start the same iteration at the average of the two values. This should settle down if our diagnosis is correct. But it does not. We see oscillations that grow until their size is as it was before.

These oscillations are stable! This is our first population model that displays persistent behavior that is not monotonic. No matter at which size you start, the species with fertility 3.1 is just a little too fertile for its own good and keeps running into overpopulation every other year. Not by much, but forever.

Judging from the previous increments of k there seems only about $k = 4$ left, but to be safe let's first try something closer to 3 first. At least it is interesting to see whether these oscillations get bigger with increasing k. They should. And how big?

■ **Exercise 1.2.12** Starting with $x = 0.66$, iterate $3.45x(1 - x)$ and $3.5x(1 - x)$ until you discern a clear pattern.