CHAPTER 2

Smooth Ergodic Theory and Nonuniformly Hyperbolic Dynamics

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HANDBOOK OF DYNAMICAL SYSTEMS, VOL. 1B
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Introduction

The goal of this chapter is to describe the contemporary status of nonuniform hyperbolicity theory. We present the core notions and results of the theory as well as discuss recent developments and some open problems. We also describe essentially all known examples of nonuniformly hyperbolic systems. Following the principles of the handbook we include informal discussions of many results and sometimes outline their proofs.

Originated in the works of Lyapunov [171] and Perron [194,195] the nonuniform hyperbolicity theory has emerged as an independent discipline in the works of Oseledets [192] and Pesin [198]. Since then it has become one of the major parts of the general dynamical systems theory and one of the main tools in studying highly sophisticated behavior associated with "deterministic chaos". We refer the reader to the article [5] by Hasselblatt and Katok in volume 1A of the handbook for a discussion on the role of nonuniform hyperbolicity theory, its relations to and interactions with other areas of dynamics. See also the article [4] by Hasselblatt in the same volume for a brief account of nonuniform hyperbolicity theory in view of the general hyperbolicity theory, and the book by Barreira and Pesin [35] for a detailed presentation of the core of the nonuniform hyperbolicity theory.

Nonuniform hyperbolicity conditions can be expressed in terms of the Lyapunov exponents. Namely, a dynamical system is nonuniformly hyperbolic if it admits an invariant measure with nonzero Lyapunov exponents almost everywhere. This provides an efficient tool in verifying the nonuniform hyperbolicity conditions and determines the importance of the nonuniform hyperbolicity theory in applications.

We emphasize that the nonuniform hyperbolicity conditions are *weak enough* not to interfere with the topology of the phase space so that any compact smooth manifold of dimension $\geqslant 2$ admits a volume-preserving C^∞ diffeomorphism which is nonuniformly hyperbolic. On the other hand, these conditions are *strong enough* to ensure that any $C^{1+\alpha}$ nonuniformly hyperbolic diffeomorphism has positive entropy with respect to any invariant *physical* measure (by physical measure we mean either a smooth measure or a Sinai–Ruelle–Bowen (SRB) measure). In addition, any ergodic component has positive measure and up to a cyclic permutation the restriction of the map to this component is Bernoulli. Similar results hold for systems with continuous time.

It is conjectured that dynamical systems of class $C^{1+\alpha}$ with nonzero Lyapunov exponents preserving a given smooth measure are typical in some sense. This remains one of the major open problems in the field and its affirmative solution would greatly benefit and boost the applications of the nonuniform hyperbolicity theory. We stress that the systems under consideration should be of class $C^{1+\alpha}$ for some $\alpha > 0$: not only the nonuniform hyperbolicity theory for C^1 systems is substantially less interesting but one should also expect a "typical" C^1 map to have some zero Lyapunov exponents (unless the map is Anosov).

In this chapter we give a detailed account of the topics mentioned above as well as many others. Among them are:

(1) stable manifold theory (including the construction of local and global stable and unstable manifolds and their absolute continuity);
(2) local ergodicity problem (i.e., finding conditions which guarantee that every ergodic component of positive measure is open $(\mathrm{mod}\, 0)$);

(3) description of the topological properties of systems with nonzero Lyapunov exponents (including the density of periodic orbits, the closing and shadowing properties, and the approximation by horseshoes); and

(4) computation of the dimension and the entropy of arbitrary hyperbolic measures.

We also describe some methods which allow one to establish that a given system has nonzero Lyapunov exponents (for example, SRB-measures) or to construct a hyperbolic measure with "good" ergodic properties (for example, the Markov extension approach). Finally, we outline a version of nonuniform hyperbolicity theory for systems with singularities (including billiards).

The nonuniform hyperbolicity theory covers an enormous area of dynamics and despite the scope of this survey there are several topics not covered or barely mentioned. Among them are nonuniformly hyperbolic one-dimensional transformations, random dynamical systems with nonzero Lyapunov exponents, billiards and related systems (for example, systems of hard balls), and numerical computation of Lyapunov exponents. For more information on these topics we refer the reader to the articles in the handbook [2,3,7–9]. Here the reader finds the ergodic theory of random transformations [8,3] (including a "random" version of Pesin's entropy formula in [8]), nonuniform one-dimensional dynamics [10,7], ergodic properties and decay of correlations for nonuniformly expanding maps [10], the dynamics of geodesic flows on compact manifolds of nonpositive curvature [9], homoclinic bifurcations and dominated splitting [11] and dynamics of partially hyperbolic systems with nonzero Lyapunov exponents [6]. Last but no least, we would like to mention the article [2] on the Teichmüller geodesic flows showing in particular, that the Kontsevich–Zorich cocycle over the Teichmüller flow is nonuniformly hyperbolic [99].

Although we included comments of historical nature concerning some main notions and basic results, the chapter is not meant to present a complete historical account of the field.

1. Lyapunov exponents of dynamical systems

Let $f^t : M \to M$ be a dynamical system with discrete time, $t \in \mathbb{Z}$, or continuous time, $t \in \mathbb{R}$, of a smooth Riemannian manifold M. Given a point $x \in M$, consider the family of linear maps $\{d_x f^t\}$ which is called *the system in variations* along the trajectory $f^t(x)$. It turns out that for a "typical" trajectory one can obtain a sufficiently complete information on stability of the trajectory based on the information on the asymptotic stability of the "zero solution" of the system in variations.

In order to characterize the asymptotic stability of the "zero solution", given a vector $v \in T_x M$, define the *Lyapunov exponent* of v at x by

$$\chi^+(x, v) = \overline{\lim_{t \to +\infty}} \frac{1}{t} \log \| d_x f^t v \|.$$

For every $\varepsilon > 0$ there exists $C = C(v, \varepsilon) > 0$ such that if $t \geqslant 0$ then

$$\| d_x f^t v \| \leqslant C e^{(\chi^+(x,v)+\varepsilon)t} \|v\|.$$

The Lyapunov exponent possesses the following basic properties: