CHAPTER 14

Global Attractors in PDE

A.V. Babin

Department of Mathematics, University of California, Irvine, CA 92697-3875, USA
E-mail: ababine@math.uci.edu

Contents

HANDBOOK OF DYNAMICAL SYSTEMS, VOL. 1B
Edited by B. Hasselblatt and A. Katok

0. Introduction

Partial differential equations and systems describe evolution of time-dependent functions and vector fields $u(x,t)$ where x is a spatial variable and t is time. We consider $u(x,t)$ with a fixed t as an element of a function space E and obtain a vector-function $u(t)$. Therefore, a partial differential equation or system can be written in the form

$$\partial_t u = \mathcal{F}\big(u(t)\big), \tag{1}$$

where the operator $\mathcal{F}(u)$ includes partial derivatives of u with respect to spatial variables $x = (x_1, \dots, x_n)$. This equation looks like an ordinary differential equation and one may try to use methods from the theory of finite-dimensional dynamical systems to study the dynamics generated by (1). Dynamics can be studied locally and globally. The local theory of equilibria, periodic solutions and their perturbations is very rich and includes their stability, bifurcations, theory of local invariant manifolds through them (see [99,203,230, 291,344,360]). Here we mostly consider global aspects of dynamics.

The dynamics generated by (1) with initial data in a function space E can be described by the solution semigroup

$$S_t : u(0) \mapsto u(t)$$

that acts in the space E. When $\mathcal{F}(u(\cdot))$ does not depend on t explicitly, the solution operators S_t satisfy the semigroup identity

$$S_{t+\tau} = S_t S_\tau, \quad t \geqslant 0,\ \tau \geqslant 0, \qquad S_0 = 1. \tag{2}$$

The long-time behavior of solutions of such equations can be adequately described in terms of global attractors of the equations. In many problems the influence of initial data has vanished after a long time has elapsed, therefore permanent regimes are of importance. The simplest permanent regimes are described by time-independent functions that are solutions of the equation $\mathcal{F}(u) = 0$. Such regimes are important but very special and it is widely believed that time-dependent permanent regimes are of importance, in particular they describe turbulence in hydrodynamics (see [102,341]). Time-dependent regimes may include time-periodic, time quasiperiodic and chaotic regimes; their common feature is that they are defined for all times, positive and negative. A mathematically rigorous description of such regimes and related questions of asymptotic behavior and stability is given by the theory of attractors. The theory of global attractors of PDE is developed in works of many mathematicians, see the list of references and in particular the books [55, 98,209,270,338,353,363,371] and references therein. Here we give a brief sketch of basic ideas, approaches and directions of research in this field. We also try to complement recent reviews [331] and [336] on related subjects from this series.

The central concept of the theory we discuss here is a global attractor. Since the terminology used in the theory of global attractors of PDE was changing with time we give a brief review of the history of related concepts. A discussion of the concept of an attractor in the theory of finite-dimensional dynamical systems is given by Milnor [308]. Usually an attractor of a semigroup (a semiflow in different terminology) is understood as an invariant set that attracts its neighborhood, it equals the omega-limit set of a neighborhood of the attractor (see [308] for different variants of this definition). Here we call such an attractor a local attractor. A dynamical system may have several local attractors, for example several stable equilibria or stable periodic solutions with different domains of attraction. In the dynamical systems generated by PDE local attractors are often considered, see [102, 230,291,341,378]. Sometimes a smaller attractor is considered, namely a set which attracts most of the points of the neighborhood, such an object is called a minimal attractor by Milnor [308], where the reader can find exact definitions. Before 1982 in the research on global dynamics of PDE, in particular in the works of Ladyzhenskaya [261,262], Foias and Temam [181], Henry [230], the attracting sets were presented as omega-limit sets of a large ball and characterized as maximal invariant bounded sets. An absorbing ball in connection with a description of the long-time dynamics of the two-dimensional Navier–Stokes system was found by Foias and Prodi [175]. The invariant set that is the omega-limit set of an absorbing ball was constructed by Ladyzhenskaya [261,262] for the two-dimensional Navier–Stokes system. One of results of [37,40] is that the invariant set constructed by Ladyzhenskaya is the global attractor in the modern terminology, namely it attracts all bounded sets in the norm-induced topology of the energy space. The seminal work of Ladyzhenskaya [262] is the first work where a global attractor of a PDE was constructed and its important properties described; in particular, the invertibility of dynamics on the attractor was proven. Ladyzhenskaya [262] also proved that a trajectory on the attractor is uniquely determined by its finite-dimensional projection, this theorem is the first in the important direction of research of finite-dimensionality of attractors of PDE; the research was continued by Mallet-Paret [286], Foias and Temam [181], Mañe [289], Foias, Temam, Manley and Treve [172], Babin and Vishik [39,42] and in many subsequent papers; for more details and references see Section 2.1.

Dynamical systems generated by PDE have their specifics. The description of dynamics usually is given in terms of inequalities that are formulated in terms of function norms, this makes them uniform in corresponding normed spaces; the inequalities describe uniformly behavior of solutions with initial data from a bounded set in such a space. A natural description of dynamics should take into account these features. The following definition of a maximal attractor in terms of attraction of all bounded sets was given and was used as a basis for a systematic approach to the study of global dynamics of parabolic, damped hyperbolic equations and the Navier–Stokes system in a series of papers of Babin and Vishik published in 1982–1983 [37,38,40,39,42] and in many subsequent papers. In these works the existence of maximal attractors was proven for general multidimensional parabolic systems, two-dimensional Navier–Stokes system and damped wave equations; the basic properties of the attractors were described; in particular, upper and lower estimates of the Hausdorff dimension of attractors were obtained and a regular structure of attractors for parabolic and hyperbolic equations with a global Lyapunov function was described. We quote in the introduction the definition from [39,42], the earlier definitions in [37,38,40] did not include the closedness (or compactness) as a requirement.

DEFINITION. A *maximal attractor* of a semigroup $\{S_t\}$ in a Banach space E is a bounded closed set $\mathcal{A}$ with the following two properties:

(i) $\mathcal{A}$ is invariant, that is $S_t\mathcal{A} = \mathcal{A}$ for all $t \geqslant 0$;
(ii) $\mathcal{A}$ attracts all bounded sets in E, that is $\delta_E(S_t B, \mathcal{A}) \to 0$ as $t \to \infty$ for every bounded set B.

This definition explicitly describes the domain of attraction, that is the whole Banach space E and, more important, explicitly specifies the attraction of $S_t u_0$ to $\mathcal{A}$. Namely, the attraction is assumed to be uniform with respect to a bounded $u_0 \in B$. Compared with the concept of a maximal invariant set that was used before in the dynamical theory of PDE this definition explicitly includes the topology of the attraction. This distinction is important in the infinite-dimensional case when the same space may be endowed with two non-equivalent topologies, for example the norm-induced and the weak topology of a Hilbert space. The maximal invariant set can be the same, but the attraction is understood in different ways and this difference is a major point of research, especially when the dynamics generated by equations in unbounded domains and damped hyperbolic problems is considered; very often the same set with the attraction in the weak topology is called a weak attractor. Before 1982–1983 in the literature on dynamical properties of PDE the attractors were considered (as omega-limit sets) but the attraction as such was not discussed.

In addition to properties of dynamics in PDE mentioned above there is the following motivation for this definition. Firstly, the maximal attractor is determined uniquely by the semigroup $\{S_t\}$, that is by the operator $\mathcal{F}$ in (1) and by the space E. Secondly, the definition does not include a specific construction of the attractor.

After 1983 the above definition of a maximal attractor or its minor variations became a standard definition in the theory of global attractors of PDE (see [55,363,209,101,270, 98,353,336] and references therein) but the name *global attractor* is now used more often. Sometimes this object is called a *universal attractor* (see [363]). We originally used the term maximal attractor to point out that the domain of attraction is maximal (namely the whole space) and that it is a maximal invariant set. Note that under natural assumptions the maximal attractor is a maximal invariant bounded set and a minimal closed set that attracts all bounded sets; the latter property is not in a perfect match with the name maximal attractor, but wise people say that nothing is perfect. The term *minimal closed B-global attractor* used by Ladyzhenskaya [270] for the same object is very precise but seems to be too long. One has to take into account that originally in the theory of infinite-dimensional dynamical systems the definition of a global attractor given in [212] did not include the attraction of bounded sets, and the global attractor was defined as a set that attracted $S_t u_0$ for all $u_0 \in E$; this terminology was used until 1984, see [214, p. 46]. Note that a set which was called a global attractor in the old terminology is usually smaller than the maximal attractor (or the global attractor in the modern terminology). By 1981 the general theory of maximal invariant sets of infinite-dimensional semigroups was developed by Billotti and La Salle [66], Hale, La Salle and Slemrod [212], Massatt [292,293]. Important concepts of asymptotically smooth semigroups were introduced by Hale, La Salle and Slemrod [212] and existence of maximal invariant sets of asymptotically smooth semigroups was proved; non-trivial sufficient conditions for the asymptotic smoothness were found; relations between different concepts of attraction were studied; see [206,208,209,336] for details and references. This theory in particular includes theorems on existence of maximal bounded invariant sets that attract all bounded sets, see [214]. One has to note though that before

1984 the attraction of bounded sets in the literature on abstract semigroups in infinite-dimensional spaces was considered among other properties such as attraction of points, attraction of compact sets and their neighborhoods and was not a subject of special interest (see, for example, [207, Chapter 4], [214]). The main application of the general theory was the dynamics of retarded functional differential equations, we could not find in the literature on Partial Differential Equations published before 1982 a paper where a theorem on attraction of every bounded set to an attractor of an equation with partial derivatives was formulated or proved.

In this review we try to pay attention to the aspects of the dynamics of PDE which distinguish this subject from the theory of finite-dimensional dynamical systems and from the abstract theory of infinite-dimensional dynamical systems.

The theory of infinite-dimensional systems generated by PDE includes technical complications that are absent in the finite-dimensional theory:

- Semigroup operators S_t often are defined only for $t \geqslant 0$ and cannot be extended for $-\infty < t < \infty$.
- Infinite-dimensional function spaces are not locally compact.
- Dynamics in infinite-dimensional spaces for given initial data as a rule does not allow an explicit description, therefore only a collective description is available, usually in terms of inequalities.
- Solutions with bounded energy can blow-up in a finite time.
- Uniqueness of solutions may be difficult to establish (3D Navier–Stokes system).
- The dependence on initial data may be non-smooth even when non-linear operators are polynomial thanks to infinite-dimensional effects (strongly non-linear monotonic parabolic equations).

More importantly, the dynamics generated by PDE has completely new features:

- Dimension of the global attractor can be considered as a large parameter, this allows to study the asymptotic behavior of the dimension.
- The spatial variables allow one to classify functions from invariant sets according to their geometric properties:
  - (i) number of zeros;
  - (ii) homotopy type;
  - (iii) symmetry properties.
- Interaction of spatial and temporal behavior (dependence of the dimension of the attractor and the fragmentation complexity of the attractor on the volume of the spatial domain).

Therefore, the central problems studied in the theory of global attractors of PDE include:

- Reduction in some sense of infinite-dimensional systems to finite-dimensional.
- Characterization of the attraction in different topologies, exponential attraction, tracking property.
- Interconnection of spatial properties of solutions and their dynamical properties.
- Expression of characteristics of attractors in terms of physical parameters of the problems.
- Relation of the properties of dynamics (for example, the existence of a global attractor) with the number of spatial variables and the growth of non-linearities.

One has to take into account that there are obvious similarities between the infinite-dimensional and finite-dimensional cases. For example, the construction of a global attractor as an omega-limit set works in both cases. The theory of local invariant manifolds and foliations is similar to the finite-dimensional theory. Though the semigroups generated by parabolic operators are not invertible (cannot be extended to negative times) the technical difficulties that arise in many cases can be solved and do not lead to significant differences.

We pay here more attention to the aspects of the theory which are specific to the infinite-dimensional case. There are completely new phenomena, for example, the dimension of the global attractors tends to infinity when the viscosity tends to zero; such behavior and its asymptotics makes sense only in an infinite-dimensional situation. Another phenomenon that has no simple analogues in the finite-dimensional case is the presence of a spatial variable in addition to the time variable. Relations between spatial and time variables manifest themselves most clearly in the case of an unbounded or a very large domain, for example the growth of the dimension of attractor and its fragmentation complexity when the domain increases, or the trivialization of dynamics on the attractor of the Navier–Stokes system in unbounded channels near spatial infinity. Many aspects of the theory of attractors are important for applications, in particular to geophysics and meteorology (see [279,280, 278]). In particular, the dimension of attractor estimates the number of degrees of freedom of the dynamical system which describes long time behavior of a physical system. A global attractor also contains all the information on the instability of the dynamical system (see [55]).

The purpose of this chapter is to give a sketch of the core of the classical theory of attractors with a minimum of technicalities and to point to major directions in which the theory develops. We do not intend to give the most general results, but rather we want to show the ideas in the simplest possible way. We prefer to present results with simple formulations rather than the most general results and give references to the literature for possible generalizations. We do not give here detailed proofs; if the formulations of results are very technical, we refer to original papers for details. Since this review reflects scientific interests of the author, inevitably not all directions in the theory of global attractors of PDE are represented with the same degree of detail. The author apologizes that many interesting papers are not discussed in this review.

1. Global attractors of semigroups

Here we discuss basic concepts related to dynamics in infinite-dimensional spaces.

1.1. *Basic definitions and existence of attractors*

Absorption and attraction. Let E be a complete metric space with distance $\rho(x_1, x_2)$ and a semigroup of (non-linear) operators $\{S_t, t \geqslant 0\}$ act in E:

$$S_t : E \to E, \quad t \geqslant 0.$$