CHAPTER 8

# On the Lyapunov Exponents of the Kontsevich–Zorich Cocycle

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## 1. Introduction

The Kontsevich–Zorich cocycle, introduced in [25], is a cocycle over the Teichmüller flow on the moduli space of holomorphic (quadratic) differentials. The study of the dynamics of this cocycle, in particular of its Lyapunov structure, has important applications to the ergodic theory of interval exchange transformations (i.e.t.'s) and related systems such as measured foliations, flows on *translation surfaces* and rational polygonal billiards (see the article by H. Masur [5] in this handbook). The Kontsevich–Zorich cocycle is a continuous-time version of a cocycle introduced by G. Rauzy [35] as a "continued fractions algorithm" for i.e.t.'s and later studied by W. Veech, in his work on the unique ergodicity of the generic i.e.t. [38], and A. Zorich [45,46] among others.

### 1.1. *Deviation of ergodic averages and other applications*

Zorich (see [44,46,47]) made the key discovery that typical trajectories of generic (orientable) measured foliations on surfaces of higher genus (or equivalently of generic i.e.t.'s with at least 4 intervals) deviate from the mean according to a power law with exponents determined by the Lyapunov exponents of the cocycle.

In [45] he began a systematic study of the Lyapunov spectrum of the cocycle and conjectured, on the basis of careful numerical experiments, that all of its Lyapunov exponents are non-zero and simple. He also observed that, as a consequence of the close connection between the cocycle and the Teichmüller geodesic flow, the simplicity of the top exponent, sometimes called the *spectral gap* property, is equivalent to the (non-uniform) hyperbolicity of the Teichmüller flow, which had been proved earlier by W. Veech [40].

The applications of the Kontsevich–Zorich cocycle to the dynamics of i.e.t.'s and related systems are not limited to the deviation of ergodic averages. The spectral gap property of the cocycle also plays an important role in recent results of Marmi, Moussa and Yoccoz [27,28] on the *cohomological equation* for generic i.e.t.'s, which improve on previous work of the author [19].

In a different direction, A. Avila and the author [7] have recently shown that the positivity of the second exponent (for surfaces of higher genus) implies that almost every i.e.t. which is not a rotation is weakly mixing and that the generic directional flow on the generic translation surface of higher genus is weakly mixing as well. This result answers in the affirmative a longstanding conjecture on the dynamics of i.e.t.'s. Special cases of the conjecture were earlier settled by A. Katok and A. Stepin [24] (for i.e.t.'s on 3 intervals) and W. Veech [39] (for i.e.t.'s on any number of intervals, but with special combinatorics).

### 1.2. *Renormalization for parabolic systems*

The role of the Kontsevich–Zorich cocycle can be explained by the somewhat vague observation that it provides a *renormalization dynamics* for i.e.t.'s (and related systems). Such systems provide fundamental examples of *parabolic* dynamics, which by definition is characterized by sub-exponential (polynomial) divergence of nearby orbits.