CHAPTER 1

Partially Hyperbolic Dynamical Systems

Boris Hasselblatt

Department of Mathematics, Tufts University, Medford, MA 02144, USA
E-mail: boris.hasselblatt@tufts.edu
url: http://www.tufts.edu/~bhasselb

Yakov Pesin

Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA
E-mail: pesin@math.psu.edu
url: http://www.math.psu.edu/pesin/

Contents

HANDBOOK OF DYNAMICAL SYSTEMS, VOL. 1B
Edited by B. Hasselblatt and A. Katok

1. Introduction

1.1. *Motivation*

1.1.1. *Smooth ergodic systems* The flows and maps that arise from equations of motion in classical mechanics preserve volume on the phase space, and their study led to the development of ergodic theory.

In statistical physics, the Boltzmann–Maxwell ergodic hypothesis, designed to help describe equilibrium and nonequilibrium systems of many particles, prompted a search for ergodic mechanical systems. In geometry, the quest for ergodicity led to the study of geodesic flows on negatively curved manifolds, where Eberhard Hopf provided the first and still only argument to establish ergodicity in the case of nonconstantly negatively curved surfaces [57]. Anosov and Sinai, in their aptly entitled work "Some smooth Ergodic Systems" [10] proved ergodicity of geodesic flows on negatively curved manifolds of any dimension.

With the development of the modern theory of dynamical systems and the availability of the Birkhoff ergodic theorem the impetus to find ergodic dynamical systems and to establish their prevalence grew stronger. Birkhoff conjectured that volume-preserving homeomorphisms of a compact manifold are generically ergodic.

1.1.2. *Hyperbolicity* The latter 1960s saw a confluence of the investigation of ergodic properties with the Smale program of studying structural stability, or, more broadly, the understanding of the orbit structure of generic diffeomorphisms. The aim of classifying (possibly generic) dynamical systems has not been realized, and there are differing views of whether it will be. Current efforts in this direction are related to the Palis conjecture (see [4]). A promising step towards understanding generic smooth systems would clearly be an understanding of structurally stable ones, and one of the high points in the theory of smooth dynamical systems is that this has been achieved: Structural stability has been found to characterize hyperbolic dynamical systems [2].

Structural stability implies that all topological properties of the orbit structure are robust. Of these, topological transitivity has a particularly natural measurable analog, namely, ergodicity. On one hand, then, robust topological transitivity of hyperbolic dynamical systems motivated the search for broader classes of dynamical systems that are robustly transitive [29]. On the other hand, this, and the fact that volume-preserving hyperbolic dynamical systems are ergodic (with respect to volume) may have led Pugh and Shub to pose a question at the end of [56] that amounts to asking whether ergodic toral automorphisms are *stably ergodic*, i.e., whether all their volume-preserving C^1 perturbations are ergodic. They later conjectured that stable ergodicity is open and dense among volume-preserving partially hyperbolic C^2 diffeomorphisms of a compact manifold.

1.1.3. *Partial hyperbolicity* In this chapter we aim to give an account of significant results about partially hyperbolic systems. The pervasive guiding principle in this theory is that hyperbolicity in the system provides the mechanism that produces complicated dynamics in both the topological and statistical sense, and that, with respect to ergodic properties,