CHAPTER 15

Hamiltonian PDEs

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1. Introduction

In this work we discuss qualitative properties of solutions for Hamiltonian partial differential equations in the finite volume case. That is, when the space-variable x belongs to a finite domain and appropriate boundary conditions are specified on the domain's boundary (or x belongs to the whole space, but the equation contains a potential term, where the potential growths to infinity as $|x| \to \infty$, cf. below Example 5.5 in Section 5.2). Most of these properties have analogies in the classical finite-dimensional Hamiltonian mechanics. In the infinite-volume case properties of the equations become rather different due to the phenomenon of radiation, and we do not touch them here.

Our bibliography is by no means complete.

NOTATION. By $\mathbb{T}^n$ we denote the torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ and write $\mathbb{T}^1 = S^1$; by $\mathbb{R}^n_+$—the open positive octant in $\mathbb{R}^n$; by $\mathbb{Z}_0$—the set of non-zero integers. By $B_\delta(x; X)$ we denote an open δ-ball in a space X, centred at $x \in X$. Abusing notation, we denote by x both the space-variable and an element of an abstract Banach space X. For an invertible linear operator J we set $\bar{J} = -J^{-1}$. The Lipschitz norm of a map f from a metric space M to a Banach space is defined as $\sup_{m\in M} \|f(m)\| + \sup_{m_1\neq m_2} \frac{\|f(m_1)-f(m_2)\|}{\operatorname{dist}(m_1,m_2)}$.

2. Symplectic Hilbert scales and Hamiltonian equations

2.1. *Hilbert scales and their morphisms*

Let X be a real Hilbert space with a scalar product $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_X$ and a Hilbert basis $\{\varphi_k \mid k \in \widetilde{\mathbb{Z}}\}$, where $\widetilde{\mathbb{Z}}$ is a countable subset of some $\mathbb{Z}^n$. Let us take a positive sequence $\{\theta_k \mid k \in \widetilde{\mathbb{Z}}\}$ which goes to infinity with k. For any s we define X_s as a Hilbert space with the Hilbert basis $\{\varphi_k\theta_k^{-s} \mid k \in \widetilde{\mathbb{Z}}\}$. By $\|\cdot\|_s$ and $\langle\cdot,\cdot\rangle_s$ we denote the norm and the scalar product in X_s (in particular, $X_0 = X$ and $\langle\cdot,\cdot\rangle_0 = \langle\cdot,\cdot\rangle$). The totality $\{X_s\}$ is called a *Hilbert scale*, the basis $\{\varphi_k\}$—the *basis of the scale* and the scalar product $\langle\cdot,\cdot\rangle$—the *basic scalar product of the scale*.

A Hilbert scale may be continuous or discrete, depending on whether $s \in \mathbb{R}$ or $s \in \mathbb{Z}$. The objects we define below and the theorems we discuss are valid in both cases.

A Hilbert scale $\{X_s\}$ possesses the following properties:

(1) X_s is compactly embedded in X_r if $s > r$ and is dense there;

(2) the spaces X_s and X_{-s} are conjugated with respect to the scalar product $\langle\cdot,\cdot\rangle$. That is, for any $u \in X_s \cap X_0$ we have

$$\|u\|_s = \sup\big\{\langle u, u'\rangle \mid u' \in X_{-s} \cap X_0, \|u'\|_{-s} = 1\big\};$$

(3) the norms $\|\cdot\|_s$ satisfy the interpolation inequality; linear operators in the spaces X_s satisfy the interpolation theorem.

Concerning these and other properties of the scales see [77] and [59].

For a scale $\{X_s\}$ we denote by $X_{-\infty}$ and X_∞ the linear spaces $X_{-\infty} = \bigcup X_s$ and $X_\infty = \bigcap X_s$.

Scales of Sobolev functions are the most important for this work: