CHAPTER 5

Random Dynamics

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Introduction

Since the time of Newton it became customary to describe the law of motion of a mechanical system by a solution $X(t)$ of an ordinary differential equation with a given initial condition $X(0) = x$. The dynamical systems ideology developed only in the 20th century suggested to look at the evolution of the whole phase space of initial conditions (and not only of a specific x) under the action of an appropriate group (or semigroup) of transformations F^t, called a flow in view of the natural analogy with hydrodynamics, so that a solution $X(t)$ with an initial condition x can be written as $F^t x$. Among early explicit manifestations of this approach was the celebrated Poincaré recurrence theorem whose statement concerns only almost all initial conditions and it has nothing to say about a specific one.

A similar but much later development occurred with stochastic dynamics. Stochastic differential equations (SDEs) were introduced by Itô at the beginning of the 1940s giving an explicit construction of diffusion processes which were studied in the 1930s by Kolmogorov via partial differential equations and measures in their path spaces. For about 40 years it was customary to consider (random) solutions $X(t,\omega)$ at time $t > 0$ of an SDE with a fixed initial condition $X(0,\omega) = x$ and the distribution of corresponding random paths was usually of prime interest. Around 1980 several mathematicians discovered that solutions of SDEs can also be represented in a similar to the deterministic case form $X(t,\omega) = F^t_\omega x$ where the family F^t_ω is called a stochastic flow (see [107]) and for each $t > 0$ and almost all ω it consists of diffeomorphisms.

With the development of dynamical systems in the 20th century it became increasingly clear that discretizing time and considering iterations of a single transformation is quite beneficial both as a tool to study the original flow generated by an ordinary differential equation, for instance, via the Poincaré first-return map, and as a rich source of new models which cannot appear in the continuous time (especially, ordinary differential equations) framework but provide an important insight into the dynamics which is free from continuous time technicalities. The next step is an observation that the evolution of physical systems is time dependent by its nature, and so they could be better described by compositions of different maps rather than by repeated applications of exactly the same transformation. It is natural to study such problems for typical, in some sense, sequences of maps picked at random in some stationary fashion. This leads to random transformations, i.e., to discrete time random dynamical systems (RDS).

Random transformations were discussed already in 1945 by Ulam and von Neumann [159] and few years later by Kakutani [74] in the framework of random ergodic theorems and their study continued in the 1970s in the framework of relative ergodic theory (see [157] and [109]) but all this attracted only a marginal interest. The appearance of stochastic flows as solutions of SDEs gave a substantial push to the subject and towards the end of the 1980s it became clear that powerful dynamical systems tools united with the probabilistic machinery can produce a scope of results which comprises now the theory of RDS. Emergence of additional structures in SDEs motivated probabilists to have a close look at the theory of smooth dynamical systems. This brought to this subject such notions as Lyapunov exponents, invariant manifolds, bifurcations, etc., which had to be adapted to the random diffeomorphisms setup. Moreover, an introduction of invariant measures of random transformations enables us to speak about such notions as the (relative) entropy, the variational principle, equilibrium states, and the thermodynamic formalism which were developed in the deterministic case in the second half of the 20th century. The probabilistic state of mind requires here to assume as little as possible about the stochasticity which drives random transformations unlike the approach in the classical ergodic theory where all measure (probability) spaces are usually assumed to be separable.

During last 20 years a lot of work was done on various aspects of RDS and some of it is presented in 5 books [82,118,47,7,49] written on this subject. The theory of RDS found its applications in statistical physics (see [152]), economics (see [155]), meteorology (see [56]) and in other fields. In this survey we describe several important parts of ergodic theory of RDS but we do not try to fulfil an impossible task to cover everything that was done in this subject. This survey consists of 5 sections among which 4 sections exhibit the theory of RDS and Section 5 deals with random perturbations of dynamical systems. Section 1 deals with the general ergodic theory and the topological dynamics of random transformations. The general setup of random transformations together with notations we use in this survey are introduced in Section 1.1 which contains basic results about the measure-theoretic (metric) entropy and generators for random transformations. Section 2 deals with constructions of random stable and unstable manifolds for RDS while Section 3 exhibits results about relations between Lyapunov exponents and the (relative) entropy such as Ruelle's inequality and Pesin's formula for RDS. In short, Sections 2 and 3 describe results which comprise what can be called as Pesin's theory for random diffeomorphisms and endomorphisms whose original deterministic version is exhibited in the article by Barreira and Pesin [1] in this volume. Section 4 exhibits the scope of results related to or relying upon the thermodynamic formalism ideology and constructions adapted to random transformations.

Section 5 about random perturbations of dynamical systems stands quite apart from other sections. The reason for its inclusion to this survey is two-fold. First, some popular models of random perturbations, where we apply at random small perturbations of a given map, lead to random transformations. Secondly, the study of both RDS and random perturbations are motivated to some extent by an attempt to understand various stability properties of dynamical systems. The first paper [135] which rises the problem of stability of dynamical systems under random perturbations appeared already in 1933 but until the 1960s this question had not attracted substantial attention. At that time random perturbations only of dynamical systems with simple dynamics were studied (see [80] and [164]) and only in the 1970s the most interesting case of systems with complicated (chaotic) dynamics had been dealt with (see [81]). Various probabilistic results on diffusion perturbations of systems with simple dynamics can be found in [64]. On the other hand, random perturbations of chaotic dynamical systems are described in [83] (see also [27]). Since then new methods and results have appeared and we will describe also some recent results concerning random perturbations of certain types of nonuniformly hyperbolic systems. We will see also how random perturbations can serve as a tool in computations of chaotic dynamical systems on a computer which, in fact, goes back to Ulam [158].

Among main topics related to RDS which are not covered by this survey are: stochastic bifurcations theory which is not yet complete but some parts of it can be found in [7], topological classification of random cocycles which is described in [47], and infinite-dimensional RDS which play an important role in various models described by partial