CHAPTER 3

Stochastic-Like Behaviour in Nonuniformly Expanding Maps

Stefano Luzzatto

Mathematics Department, Imperial College, London, UK
E-mail: stefano.luzzatto@imperial.ac.uk
url: htttp://www.ma.ic.ac.uk/~luzzatto

Contents

1. Introduction . . . 267
   1.1. Hyperbolic dynamics . . . 267
   1.2. Expanding dynamics . . . 267
   1.3. General overview of the notes . . . 269
2. Basic definitions . . . 269
   2.1. Invariant measures . . . 270
   2.2. Ergodicity . . . 270
   2.3. Absolute continuity . . . 272
   2.4. Mixing . . . 273
   2.5. Decay of correlations . . . 274
3. Markov structures . . . 275
   3.1. The invariant measure for F . . . 279
   3.2. The invariant measure for f . . . 282
   3.3. Expansion and distortion estimates . . . 284
4. Uniformly expanding maps . . . 285
   4.1. The smooth/Markov case . . . 285
   4.2. The non-Markov case . . . 286
5. Almost uniformly expanding maps . . . 287
6. One-dimensional maps with critical points . . . 289
   6.1. Unimodal maps . . . 290
   6.2. Multimodal maps . . . 291
   6.3. Benedicks–Carleson maps . . . 292
   6.4. Expansion outside Δ . . . 294
   6.5. Shadowing the critical orbit . . . 294
   6.6. The escape partition . . . 296
   6.7. The return partition . . . 299
7. General theory of nonuniformly expanding maps . . . 301
   7.1. Measuring the degree of nonuniformity . . . 301

HANDBOOK OF DYNAMICAL SYSTEMS, VOL. 1B
Edited by B. Hasselblatt and A. Katok

7.2. Viana maps . . . 303
8. Existence of nonuniformly expanding maps . . . 304
8.1. The definition of Ω^* . . . 306
8.2. Expansion outside the critical neighbourhood . . . 309
8.3. The binding period . . . 309
8.4. Positive exponents in dynamical space . . . 311
8.5. Positive measure in parameter space . . . 312
9. Conclusion . . . 315
9.1. What causes slow decay of correlations? . . . 315
9.2. Stability . . . 316
9.3. Nonuniform hyperbolicity and induced Markov maps . . . 317
9.4. Verifying nonuniform expansivity . . . 318
Acknowledgements . . . 320
References . . . 320

1. Introduction

1.1. *Hyperbolic dynamics*

A feature of many real-life phenomena in areas as diverse as physics, biology, finance, economics, and many others, is the *random-like* behaviour of processes which nevertheless are clearly *deterministic*. On the level of applications this dual aspect has proved very problematic. Specific mathematical models tend to be developed either on the basis that the process is deterministic, in which case sophisticated numerical techniques can be used to attempt to understand and predict the evolution, or that it is random, in which case probability theory is used to model the process. Both approaches lose sight of what is probably the most important and significant characteristic of the system which is precisely that it is deterministic *and* has random-like behaviour. The theory of Dynamical Systems has contributed a phenomenal amount of work showing that it is perfectly natural for completely deterministic systems to behave in a very random-like way and achieving a quite remarkable understanding of the mechanisms by which this occurs.

We shall assume that the state space can be represented by a compact Riemannian manifold M and that the evolution of the process is given by a map $f: M \to M$ which is piecewise differentiable. Following an approach which goes back at least to the first half of the 20th century, we shall discuss how certain statistical properties can be deduced from geometrical assumptions on f formulated explicitly in terms of "*hyperbolicity*" assumptions on the *derivative map* Df of f. This is often referred to as *Hyperbolic Dynamics* or *Smooth Ergodic Theory*. The basic strategy is to construct certain geometrical structures (invariant manifolds, partitions) which imply some statistical/probabilistic properties of the dynamics. A striking and pioneering example of this is the work of Hopf on the ergodicity of geodesic flows on manifolds of negative curvature [80]. The subject has grown enormously since then to become of one of the key areas in the modern theory of Dynamical Systems. This is reflected in the present handbook in which several surveys, see [1–3,5, 7–9] address different facets of the theory in the case of *diffeomorphisms*.

The main focus of these notes will be on the analogous theory for *endomorphisms*. In this case the hyperbolicity conditions reduce to *expansivity* conditions. We shall concentrate here on three particular types of results about expanding maps: the existence of Markov structures, the existence of absolutely continuous invariant probability measures, and estimates on the rates of decay of correlations. See also [13] for a more detailed treatment of the theory and other results such as stochastic stability.

1.2. *Expanding dynamics*

We start with the basic definition of an expanding map.

DEFINITION 1. We say that $f: M \to M$ is (nonuniformly) *expanding* if there exists $\lambda > 0$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \left\| Df^{-1}_{f^i(x)} \right\|^{-1} > \lambda \tag{$*$}$$