CHAPTER 13

Pointwise Ergodic Theorems for Actions of Groups

Amos Nevo*

Department of Mathematics, Technion, Haifa, Israel
E-mail: anevo@tx.technion.ac.il

Contents

*The author was supported in part by ISF Grant #126-01.

HANDBOOK OF DYNAMICAL SYSTEMS, VOL. 1B
Edited by B. Hasselblatt and A. Katok

1. Introduction

In recent years a number of significant results and developments related to pointwise ergodic theorems for general measure-preserving actions of locally compact second countable (lcsc) groups have been established, including the solution of several long-standing open problems. The exposition that follows aims to survey some of these results and their proofs, and will include, in particular, an exposition of the following results.

(1) A complete solution to the ball averaging problem on Lie, and more generally lcsc, groups of polynomial volume growth. Namely, a proof that for any metric quasi-isometric to a word metric (and in particular, Riemannian metrics on nilpotent Lie groups), the normalized ball averages satisfy the pointwise ergodic theorem in L^1. This brings to a very satisfactory close a long-standing problem in ergodic theory, dating at least to Calderon's 1952 paper on groups satisfying the doubling condition.

(2) In fact, two independent solutions will be described regarding the ball averaging problem in the case of connected Lie groups with polynomial volume growth, but both have the following in common. They resolve, in particular, a long-standing conjecture in the theory of amenable groups, dating at least to F. Greenleaf's 1969 book [58]. The conjecture asserts that the sequence of powers of a neighborhood on an amenable group constitute an asymptotically invariant sequence, namely has the Følner property. This conjecture was disproved for solvable groups with exponential growth, but has been now verified for groups with polynomial growth.

(3) The pointwise ergodic theorem in L^1 for a tempered sequence of asymptotically invariant sets was established recently, improving on the case of L^2 established earlier. This result resolves the long-standing problem of constructing *some* pointwise ergodic sequence in L^1 on an *arbitrary* amenable group. The ideas of the two available proofs will be briefly described.

(4) A new and streamlined account of the classical Dunford–Zygmund method will be described. This account allows the derivation of pointwise ergodic theorems for asymptotically invariant sequence on any lcsc amenable algebraic (or Lie) group over any local field, generalizing the Greenleaf–Emerson theorem. It also allows the construction of pointwise ergodic sequences on any lcsc algebraic (or Lie) group over any local field, generalizing Templeman's theorem for lcsc connected groups.

(5) A general spectral method will be described for the derivation of pointwise ergodic theorems for ball averages on Gelfand pairs. This method will be demonstrated for the ball averages on any lcsc simple algebraic group (over any local field). Pointwise theorems will be demonstrated also for the natural singular spherical averages on some of the Gelfand pairs.

(6) The proof of a pointwise ergodic theorem for actions of the free groups, generalizing Birkhoff's and Wiener's theorems for $\mathbb{Z}$ and $\mathbb{Z}^d$ will be described, using the general spectral method referred to above.

(7) The derivation of pointwise ergodic theorems for actions of simple algebraic groups with an explicit exponentially fast rate of convergence to the ergodic mean will be described. The same result will be described also for certain discrete lattice subgroups.

(8) Some ergodic theorems for semisimple Lie groups of real rank at least two will be described, which are in marked contrast to the results that Euclidean analogs might suggest, a contrast which has its roots in the exponential volume growth on semisimple groups.

Our goal is to elucidate some of the main ideas used in the proof of the pointwise ergodic theorems alluded to above. Our account of the pointwise ergodic theorems for groups with polynomial volume growth will be quite detailed, as these results are very recent and have not appeared before elsewhere. However, in the case of the spectral method, we have specifically attempted to give an account of the proofs which is as elementary as possible and demonstrated using the simplest available examples. The motivation for these choices is that the spectral methods which we employ require considerable background in the structure theory and representation theory of semisimple Lie groups, as well as classical singular integral theory. At this time these methods are not yet part of the standard tool kit in ergodic theory, and consequently it seems appropriate to give an exposition which focuses on the ergodic theorems and explains some of the main ideas in their proof, but requires as little as possible by way of background.

We have also tried to emphasized the pertinent open problems in the theory, many of which are presented along the way.

Ergodic theorems for actions of connected Lie groups, and particularly equidistribution theorems on homogeneous spaces and moduli spaces, have been developed and used in a rapidly expanding array of applications, many of which are presented in the two volumes of the present handbook. Thus it seems reasonable to limit the scope of our discussion in the present exposition and concentrate specifically on pointwise ergodic theorems, which have not been treated elsewhere.

We must note however that even within the more limited scope of pointwise ergodic theorems for general group actions our account has some important omissions. We mention some of these below, and offer as our rationale the fact that there already exist good expositions of these topics in the literature, some of which are referred to below. These omissions includes the analytic theory of homogeneous nilpotent Lie groups, and in particular the extensive theory of convolution operators, harmonic analysis, maximal functions and pointwise convergence theorems for diverse averages on Euclidean spaces, Heisenberg (-type) groups, homogeneous nilpotent groups and harmonic AN-groups (see [141] and [38] for an introduction to some of these topics). They also include the extensive results on equidistribution on homogeneous spaces (see [41] and [138] for surveys, [57] for some new results), as well as the general theory of actions of amenable locally compact second countable (lcsc) groups (see [121], and [134,92,154] for more recent results). Another omission is the mean ergodic theorem for semisimple groups proved in [150], and other ergodic theorems on moduli spaces and their applications, which are described in detail in the present volume.

It is also natural to include in a discussion of maximal inequalities for group actions a discussion of convolution operators, particularly radial averages on general lcsc groups and their homogeneous spaces. This subject, for which the theory is very incomplete receives only very scant mention here, and we refer to [111] for a short survey and many open problems.