

CHAPTER 13

Pointwise Ergodic Theorems for Actions of Groups

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Contents

1. Introduction	873
2. Averaging along orbits in group actions	875
2.1. Averaging operators	875
2.2. Ergodic theorems	876
2.3. Maximal functions	877
2.4. A general recipe for proving pointwise ergodic theorems	879
3. Ergodic theorems for commutative groups	879
3.1. Flows of 1-parameter groups: Birkhoff's theorem	879
3.2. Flows of commutative multi-parameter groups: Wiener's theorem	881
4. Invariant metrics, volume growth, and ball averages	883
4.1. Growth type of groups	883
4.2. Invariant metrics	884
4.3. The ball averaging problem in ergodic theory	885
4.4. Exact volume growth	886
4.5. Strict volume growth	889
4.6. Balls and asymptotic invariance under translations	890
5. Pointwise ergodic theorems for groups of polynomial volume growth	893
5.1. Step I: The mean ergodic theorem	894
5.2. Step II: Pointwise convergence on a dense subspace	894
5.3. Step III: The maximal inequality for ball averages	895
5.4. Step IV: Interpolation arguments	899
5.5. Groups of polynomial volume growth: general case	899
6. Amenable groups: Følner averages and their applications	901
6.1. The transfer principle for amenable groups	901
6.2. Generalizations of the doubling condition: regular Følner sequences	906
6.3. Subsequence theorems: tempered Følner sequences	907
7. A non-commutative generalization of Wiener's theorem	909
7.1. The Dunford–Zygmund method	909
7.2. The ergodic theory of semidirect products	913
7.3. Structure theorems and ergodic theorems for amenable groups	915

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7.4. Structure theorems and ergodic theorems for non-amenable groups	917
7.5. Groups of bounded generation	920
7.6. From amenable to non-amenable groups: some open problems	922
8. Spherical averages	923
8.1. Euclidean spherical averages	924
8.2. Non-Euclidean spherical averages	928
8.3. Radial averages on free group	929
9. The spectral approach to maximal inequalities	931
9.1. Isometry groups of hyperbolic spaces	931
9.2. Commutativity of spherical averages	932
9.3. Littlewood–Paley square functions	933
9.4. Exponential volume growth: ball versus shell averages	936
9.5. Square functions and analytic interpolation	937
9.6. The L^p -theorem for sphere averages on $\text{Iso}(\mathbb{H}^n)$	938
10. Groups with commutative radial convolution structure	940
10.1. Gelfand pairs	940
10.2. Pointwise theorems for commuting averages: general method	942
10.3. Pointwise theorems and the spectral method: some open problems	944
10.4. Sphere averages on complex groups	946
10.5. Radial structure on lattice subgroups: a generalization of Birkhoff’s theorem	947
11. Actions with a spectral gap	953
11.1. Pointwise theorems with exponentially fast rate of convergence	955
11.2. The spectral transfer principle	959
11.3. Higher-rank groups and lattices	961
12. Beyond radial averages	963
12.1. Recipe for pointwise theorems with rate of convergence	963
12.2. Horospherical averages	965
12.3. Averages on discrete subgroups	967
13. Weighted averages on discrete groups and Markov operators	969
13.1. Uniform averages of powers of a Markov operator	969
13.2. Subadditive sequences of Markov operators, and maximal inequalities on hyperbolic groups	971
13.3. The powers of a self-adjoint Markov operators	972
14. Further developments	973
14.1. Some non-Euclidean phenomena in higher-rank groups	973
14.2. Best possible rate of convergence in the pointwise theorem	975
14.3. Added in proof	975
References	977

1. Introduction

In recent years a number of significant results and developments related to pointwise ergodic theorems for general measure-preserving actions of locally compact second countable (lcsc) groups have been established, including the solution of several long-standing open problems. The exposition that follows aims to survey some of these results and their proofs, and will include, in particular, an exposition of the following results.

- (1) A complete solution to the ball averaging problem on Lie, and more generally lcsc, groups of polynomial volume growth. Namely, a proof that for any metric quasi-isometric to a word metric (and in particular, Riemannian metrics on nilpotent Lie groups), the normalized ball averages satisfy the pointwise ergodic theorem in L^1 . This brings to a very satisfactory close a long-standing problem in ergodic theory, dating at least to Calderon's 1952 paper on groups satisfying the doubling condition.
- (2) In fact, two independent solutions will be described regarding the ball averaging problem in the case of connected Lie groups with polynomial volume growth, but both have the following in common. They resolve, in particular, a long-standing conjecture in the theory of amenable groups, dating at least to F. Greenleaf's 1969 book [58]. The conjecture asserts that the sequence of powers of a neighborhood on an amenable group constitute an asymptotically invariant sequence, namely has the Følner property. This conjecture was disproved for solvable groups with exponential growth, but has been now verified for groups with polynomial growth.
- (3) The pointwise ergodic theorem in L^1 for a tempered sequence of asymptotically invariant sets was established recently, improving on the case of L^2 established earlier. This result resolves the long-standing problem of constructing *some* pointwise ergodic sequence in L^1 on an *arbitrary* amenable group. The ideas of the two available proofs will be briefly described.
- (4) A new and streamlined account of the classical Dunford–Zygmund method will be described. This account allows the derivation of pointwise ergodic theorems for asymptotically invariant sequence on any lcsc amenable algebraic (or Lie) group over any local field, generalizing the Greenleaf–Emerson theorem. It also allows the construction of pointwise ergodic sequences on any lcsc algebraic (or Lie) group over any local field, generalizing Templeman's theorem for lcsc connected groups.
- (5) A general spectral method will be described for the derivation of pointwise ergodic theorems for ball averages on Gelfand pairs. This method will be demonstrated for the ball averages on any lcsc simple algebraic group (over any local field). Pointwise theorems will be demonstrated also for the natural singular spherical averages on some of the Gelfand pairs.
- (6) The proof of a pointwise ergodic theorem for actions of the free groups, generalizing Birkhoff's and Wiener's theorems for \mathbb{Z} and \mathbb{Z}^d will be described, using the general spectral method referred to above.
- (7) The derivation of pointwise ergodic theorems for actions of simple algebraic groups with an explicit exponentially fast rate of convergence to the ergodic mean will be described. The same result will be described also for certain discrete lattice subgroups.

- (8) Some ergodic theorems for semisimple Lie groups of real rank at least two will be described, which are in marked contrast to the results that Euclidean analogs might suggest, a contrast which has its roots in the exponential volume growth on semisimple groups.

Our goal is to elucidate some of the main ideas used in the proof of the pointwise ergodic theorems alluded to above. Our account of the pointwise ergodic theorems for groups with polynomial volume growth will be quite detailed, as these results are very recent and have not appeared before elsewhere. However, in the case of the spectral method, we have specifically attempted to give an account of the proofs which is as elementary as possible and demonstrated using the simplest available examples. The motivation for these choices is that the spectral methods which we employ require considerable background in the structure theory and representation theory of semisimple Lie groups, as well as classical singular integral theory. At this time these methods are not yet part of the standard tool kit in ergodic theory, and consequently it seems appropriate to give an exposition which focuses on the ergodic theorems and explains some of the main ideas in their proof, but requires as little as possible by way of background.

We have also tried to emphasize the pertinent open problems in the theory, many of which are presented along the way.

Ergodic theorems for actions of connected Lie groups, and particularly equidistribution theorems on homogeneous spaces and moduli spaces, have been developed and used in a rapidly expanding array of applications, many of which are presented in the two volumes of the present handbook. Thus it seems reasonable to limit the scope of our discussion in the present exposition and concentrate specifically on pointwise ergodic theorems, which have not been treated elsewhere.

We must note however that even within the more limited scope of pointwise ergodic theorems for general group actions our account has some important omissions. We mention some of these below, and offer as our rationale the fact that there already exist good expositions of these topics in the literature, some of which are referred to below. These omissions includes the analytic theory of homogeneous nilpotent Lie groups, and in particular the extensive theory of convolution operators, harmonic analysis, maximal functions and pointwise convergence theorems for diverse averages on Euclidean spaces, Heisenberg (-type) groups, homogeneous nilpotent groups and harmonic AN -groups (see [141] and [38] for an introduction to some of these topics). They also include the extensive results on equidistribution on homogeneous spaces (see [41] and [138] for surveys, [57] for some new results), as well as the general theory of actions of amenable locally compact second countable (lsc) groups (see [121], and [134,92,154] for more recent results). Another omission is the mean ergodic theorem for semisimple groups proved in [150], and other ergodic theorems on moduli spaces and their applications, which are described in detail in the present volume.

It is also natural to include in a discussion of maximal inequalities for group actions a discussion of convolution operators, particularly radial averages on general lsc groups and their homogeneous spaces. This subject, for which the theory is very incomplete receives only very scant mention here, and we refer to [111] for a short survey and many open problems.