

CHAPTER 4

Homoclinic Bifurcations, Dominated Splitting, and Robust Transitivity

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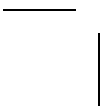
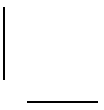
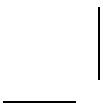
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1. Introduction

For a long time (mainly since Poincaré) it has been a goal in the theory of dynamical systems to describe the dynamics from the generic viewpoint, that is, describing the dynamics of “big sets” (residual, dense, etc.) of the space of all dynamical systems.

It was briefly thought in the sixties that this could be realized by the so-called hyperbolic systems with the assumption that the tangent bundle over the limit set $L(f)$ (see definition in Section 4.3) splits into two complementary subbundles $T_{L(f)}M = E^s \oplus E^u$ such that vectors in E^s (respectively E^u) are uniformly forward (respectively backward) contracted by the tangent map Df (see Chapter 1, Principal structures (Hasselblatt and Katok), in Volume 1A of this handbook). Under this assumption, the limit set decomposes into a finite number of transitive sets such that the asymptotic behavior of any orbit is described by the dynamics in the trajectories in those finitely many transitive sets. Moreover, this topological description allows to get the statistical behavior of the system. In other words, hyperbolic dynamics on the tangent bundle characterizes the dynamics over the manifold from a geometrical–topological and statistical point of view.

Nevertheless, uniform hyperbolicity was soon realized to be a property less universal than it was initially thought: it was shown that there are open sets in the space of dynamics which are nonhyperbolic. The initial mechanism to show this nonhyperbolic robustness (see [1,90]) was the existence of open sets of diffeomorphisms exhibiting hyperbolic periodic points of different indices (i.e. different dimension of their stable manifolds) inside a transitive set. Indeed, Shub’s construction leads to an open set of transitive diffeomorphisms on T^4 exhibiting hyperbolic periodic points of different indices. Related to this is the notion of hetero-dimensional cycle where two periodic points of different indices are linked through the intersection of their stable and unstable manifolds (notice that at least one of the intersections is nontransversal).

In all the above examples the underlying manifolds must be of dimension at least three and so the case of surfaces was still unknown at the time. It was shown through the seminal works of Newhouse (see [59,61,63]) that hyperbolicity was not dense in the space of C^r -diffeomorphisms of compact surfaces (however, let us point out that in the C^1 -topology it is still open!). The underlying mechanism here was the presence of a homoclinic tangency leading to the nowadays so-called “Newhouse phenomenon”, i.e. residual subsets of diffeomorphisms displaying infinitely many periodic attractors.

These results naturally pushed some aspects of the theory of dynamical systems in different directions:

1. The study of the dynamical phenomena obtained from homoclinic bifurcations (i.e. the unfolding of homoclinic tangencies or hetero-dimensional cycles);
2. The characterization of universal mechanisms that could lead to robustly (meaning any perturbation of the initial system) nonhyperbolic behavior;
3. The study and characterization of isolated transitive sets that remain transitive for all nearby systems (*robust transitivity*);
4. The dynamical consequences of weaker forms of hyperbolicity.

As we will show, these problems are all related to each other. In many cases, such relations provide a conceptual framework, as the hyperbolic theory did for the case of transverse homoclinic orbits.