CHAPTER 16

Extended Hamiltonian Systems

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1. Introduction

In this chapter we discuss Hamiltonian partial differential wave equations which are defined on unbounded spatial domains, a class of so-called *extended Hamiltonian systems*. The examples we consider are the nonlinear Schrödinger and Klein–Gordon equations defined on $\mathbb{R}^3$. These may be viewed as infinite-dimensional Hamiltonian systems, which have coherent solutions, e.g., spatially uniform equilibria, spatially nonuniform solitary standing waves.... Questions of interest include the dynamics in a neighborhood of these states (stability to small perturbations), stability under small Hamiltonian perturbations of the dynamical system, the behavior of solutions on short, intermediate and infinite time scales and the manner in which these coherent states participate in the structure of solutions on these time scales.

The contrast in dynamics between Hamiltonian systems of extended type and those of compact type is striking. Compact Hamiltonian systems arising, for example, from finite-dimensional Hamiltonian systems or Hamiltonian partial differential equations (PDEs) governing an evolutionary process defined on a bounded spatial domain, are systems governed by finite or infinite systems of ordinary differential equations (ODEs) with a *discrete* set of frequencies. Many fundamental phenomena and questions here involve the persistence or breakdown of regular (e.g., time periodic or quasiperiodic) solutions and their dynamical stability relative to small perturbations. A stable state of the system is one around which neighboring trajectories oscillate. KAM theory implies states persist in the presence of small Hamiltonian perturbations (structural stability) provided certain arithmetic *nonresonance* conditions on the set of frequencies of the unperturbed state hold [1,27,11,3].

In contrast, extended Hamiltonian systems arising from Hamiltonian PDEs are systems involving continuous as well as discrete spectra of frequencies. Stable states are expected to be *asymptotically stable*; states initially nearby the unperturbed state remain close and even converge to it in an appropriate metric. Since the flow is in an infinite-dimensional space, this does not contradict the Hamiltonian character of the phase flow, which in finite-dimensional spaces preserves volume. Convergence to an asymptotic state occurs through a mechanism of radiating energy to infinity. It is also possible that some states of the system are long-lived *metastable states*. These are states which persist on long time scales, but decay as $t \to \infty$. This structural instability due to Hamiltonian perturbations occurs due to nonlinearity induce resonances of states associated with discrete and continuous spectra, precisely that which is precluded in the setting of KAM theory.

2. Overview

We consider partial differential equations for which the linear part (the small amplitude limit) has spatially localized and time-periodic "bound state" solutions, which are dynamically stable. Such solutions of the linear dynamical system are associated with the discrete spectrum of linear self-adjoint operator generating the flow. Also associated with this operator, due to the unboundedness of the spatial domain, is continuous spectrum with corresponding spatially extended (nondecaying) radiation states. These bound and radiation states are central to the linear dynamics. Arbitrary finite energy initial conditions can,