MATH 312H:
FUNDAMENTAL STRUCTURES OF CONTINUOUS MATHEMATICS

SPRING 2004

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FINAL EXAMINATION

April 30, 4:15pm

Write a detailed solution for each of the problems below. After you've read and understood the problems you have four hours to do that in one stretch at any time between now and midnight on Sunday May 2. Please put written solutions into my mailbox on the second floor of the McAllister building by noon on Monday May 3.

You may use results discussed in class in your solutions. For the external material you may use your own course notes only or notes of your classmates copied before you open the problems. You may not consult with anyone.

If you have any questions please e-mail me immediately *before* you start the clock. You may call me on my cell phone after 6pm 769-9443

If you feel that you cannot do enough problems in a satisfactory fashion by the end of alloted time please contact me and we will schedule an appointment early next week for an oral discussion of the problems and related questions from the course.

Good luck!

F1. A real number t is called *transcendental* is it is not algebraic, i.e. it is not a root of a polynomial with rational coefficients.

Prove that for any transcendental number t there exist a *field*, i.e. a set of real numberss closed under addition, subtraction, multiplication and division by any non-zero element) which contains t and consists of only countably many elements

Think how elements of such a field may look

F2. Prove rigorously the *Archimedean property* of the real numbers: For any numbers $a > 0$ and b there exist a natunal number n such that $an > b$

The point of this problem is to establish this property first for rational numbers and to argue how it extends to real numbers which are defined as equivalence classes of Cauchy sequences of rational numbers.

F3. Consider the set $\mathcal{S}_n$ of all subsets of the set with n elements (n is a natural number here) For two such subsets A and B define the distance between A and B as the number of elements which belong to exaclty one of the sets (i.e. in A but not in B, or in B but not in A). Prove that $\mathcal{S}_n$ with this distance is a metric space.

Make sure sure you check all properties of metric!

F4. Consider the space $\mathcal{T}$ of quadratic polynomials P with complex coefficients, i.e. of the form $P(z) = az^2 + bz + c$ where a, b, c are complex numbers. For two such polynomials P and Q define the distance $d(P, Q)$ as $|P(1) - Q(1)| + |P(-1) - Q(-1)| + |P(i) - Q(i)|$

Prove that $\mathcal{T}$ with this distance is a metric space.

Make sure you check all properties of metric!

F5. Let X, d be a metric space. For three points $x_1, x_2, x_3 \in X$ (not necessarily all different) define the *perimeter* $p(x_1, x_2, x_3)$ as $d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1)$,

(i) Prove that perimeter does not depend on the order of points.

(ii)If X is compact prove that there exist a triple of points x_1, x_2, x_3 (not necessarily all different) of maximal perimeter, i.e. $p(x_1, x_2, x_3) \geq p(y_1, y_2, y_3)$ for any points $(y_1, y_2, y_3) \in X$.