MATH 312H

REAL NUMBERS

A *field* F is a set with two binary operations usually called *addition* and *multiplication* which satisfy the most basic properties of these two operations for numbers. Namely,

(1) $\forall\ a, b \in F$, $a + b = b + a$ (commutativity of addition),
(2) $\forall\ a, b, c \in F$, $a + (b + c) = (a + b) + c$ (associativity of addition),
(3) $\exists\ 0 \in F$ such that $\forall\ a \in F$, $0 + a = a$ (existence of zero),
(4) $\forall\ a \in F\ \ \exists -a \in F$ such that $a + (-a) = 0$ (existence of the additive inverse),
(5) $\forall\ a, b \in F$, $a \cdot b = b \cdot a$ (commutativity of multiplication)
(6) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associativity of multiplication),
(7) $\exists\ 1 \in F$ such that $\forall\ a \in F^{\times} = F - \{0\}$, $1 \cdot a = a$ (existence of identity),
(8) $\forall\ a \in F^{\times}\ \ \exists\ a^{-1} \in F^{\times}$ such that $a \cdot a^{-1} = 1$ (existence of the multiplicative inverse),
(9) $\forall\ a, b, c \in F$, $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributivity),
(10) $0 \neq 1$.

An algebraic structure with only one binary operation satisfying the properties (1) – (4) is called an *abelian (or commutative) group*. Correspondingly, F with addition is called the *additive group* of the field F, and $F^{\times}$ with multiplication is called the *multiplicative group* of the field F. An important property of a field is that it does not contain *zero–divisors*:

1. Prove that if $a, b \in F$ are such that $a \cdot b = 0$ then either $a = 0$ or $b = 0$.

2. Give examples of fields.

There three ways to define real numbers beginning with $\mathbb{Q}$:

- an explicit construction via infinite decimal fractions;
- using the *norm* on $\mathbb{Q}$: $|x|$ – the absolute value; using different norms on $\mathbb{Q}$ one obtains other interesting fields containing $\mathbb{Q}$ (p–adic numbers);
- using the notion of *order* on $\mathbb{Q}$: $x < y$ ("Dedekind's cuts"); it gives essentially the unique field of real numbers $\mathbb{R}$.

Construction of real numbers by completion of rationals

We start with natural numbers (positive integers) and introduce negative numbers and 0 to make a group of integers $\mathbb{Z}$ by addition. Then we define $\mathbb{Q}$ as the set of fractions in order to obtain a field. Thus, we have a notion of *positive* rational number: $\frac{p}{q} > 0$ if p and q are both positive or both negative. We also have a notion of *order* on $\mathbb{Q}$: $x < y$ iff $y - x > 0$.

For $x \in \mathbb{Q}$ we define the *absolute value*

$$|x| = \begin{cases} x & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -x & \text{if} \quad x < 0. \end{cases}$$

3. Prove that $|\cdot|$ is a norm on $\mathbb{Q}$, i.e.

(1) $|x| = 0$ if and only if $x = 0$
(2) $|xy| = |x|\,|y|, \quad \forall\, x, y \in \mathbb{Q}$
(3) $|x + y| \le |x| + |y|, \quad \forall\, x, y \in \mathbb{Q}$ (*triangle inequality*).

Notice that the norm $|\cdot|$ takes non-negative rational values.
4. Prove that for all $x, y \in \mathbb{Q}$ $|x \pm y| \ge ||x| - |y||$.

Let $d(x, y) = |x - y|$.

5. Prove that d is a *distance function* on $\mathbb{Q}$, i.e.

(1) $d(x, y) \ge 0$, and $d(x, u) = 0$ if and only if $x = y$
(2) $d(x, y) = d(y, x)$
(3) $|d(x, z) \le d(x, y) + d(y, z)$ (*triangle inequality*).

Now $(\mathbb{Q}, |\cdot|)$ is a *metric space*. A sequence $\{a_n\}$ in $\mathbb{Q}$ is said to be

- *bounded* if there is a constant $C > 0$ (of course, $C \in \mathbb{Q}$) such that
$$|a_n| \le C \ \ \forall n;$$
- a *null* sequence if
$$\lim_{n\to\infty} |a_n| = 0,$$
i.e., for any $\varepsilon > 0$ ($\varepsilon \in \mathbb{Q}$) there is a natural number N such that for all $n > N$ $|a_n| < \varepsilon$;
- a *Cauchy* sequence if
$$\lim_{n,m\to\infty} |a_n - a_m| = 0,$$
i.e., for any $\varepsilon > 0$ there is an N such that for all $n, m > N$ we have $|a_n - a_m| < \varepsilon$;

- *convergent* to $a \in F$ (we write $a = \lim_{n\to\infty} a_n$) if
$$\lim_{n\to\infty} |a_n - a| = 0,$$
i.e., for any $\varepsilon > 0$ there is an N such that for all $n > N$ $|a_n - a| < \varepsilon$.

6. Prove that
(1) any null sequence converges to 0,
(2) any converging sequence is a Cauchy sequence.

In particular, every null sequence is a Cauchy sequence.

7. Prove that in $\mathbb{Q}$
(1) Every Cauchy sequence is bounded.
(2) Let $\{a_n\}$ be a Cauchy sequence and $\{n_1, n_2, \dots\}$ be an increasing sequence of positive integers. If the subsequence $a_{n_1}, a_{n_2}, \dots$ is a null sequence, then $\{a_n\}$ itself is a null sequence.
(3) If $\{a_n\}$ and $\{b_n\}$ are null sequences, so is $\{a_n \pm b_n\}$, and if $\{a_n\}$ is a null sequence and $\{b_n\}$ is a bounded sequence, then $\{a_n b_n\}$ is a null sequence.

8. Prove that $|x| < 1$ if and only if $\lim_{n\to\infty} x^n = 0$.

However, $\mathbb{Q}$ **is not complete with respect to** $|\cdot|$, i.e. not every Cauchy sequence in $\mathbb{Q}$ has a limit in $\mathbb{Q}$.

9. Give an example of a Cauchy sequence of rational numbers which has no limit in $\mathbb{Q}$.

Cauchy sequences can be added, subtracted and multiplied.

10. Prove that if $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, then so are
$$\{a_n + b_n\}, \{a_n - b_n\}, \text{ and } \{a_n b_n\}.$$

Therefore the set of all Cauchy sequences in $(\mathbb{Q}, |\cdot|)$, denoted by $\{\mathbb{Q}\}$, becomes a commutative ring. Its identity element under addition is the sequence
$$\bar{0} = \{0, 0, 0, \dots\},$$
and its identity element under multiplication is the sequence
$$\bar{1} = \{1, 1, 1, \dots\}.$$
It is clear that $\{\mathbb{Q}\}$ is not a field since it contains zero divisors:
$$\{1, 0, 0, \dots\}\{0, 1, 0, 0, \dots\} = \bar{0}.$$
For every $a \in Q$ the Cauchy sequence
$$\bar{a} = \{a, a, a \dots\}$$

lies in $\{\mathbb{Q}\}$. Hence $\{\mathbb{Q}\}$ contains a subring isomorphic to $\mathbb{Q}$. Of particular importance is the set P of all null sequences. By Exercise 7(2) P is a subset of $\{\mathbb{Q}\}$. In fact, P is an *ideal* in $\{\mathbb{Q}\}$ (i.e., a subring such that for all $p \in P$ and all $a \in \mathbb{Q}$ we have $ap \in P$). Indeed, if $\{a_n\}$ and $\{b_n\}$ are in P, so is $\{a_n \pm b_n\}$, and if $\{a_n\}$ is in P and $\{b_n\}$ is a bounded sequence (in particular if it is Cauchy), then $\{a_n b_n\}$ is in P (Exercise 7(3)).

Let $\overline{\mathbb{Q}} = \{\mathbb{Q}\}/P$. Its elements are equivalence classes of Cauchy sequences in $(\mathbb{Q}, |\cdot|)$, two Cauchy sequences being *equivalent* if their difference is a null sequence. Notice that constant sequences

$$\bar{a} = \{a, a, a, \dots\},$$

where $a \in \mathbb{Q}$ belong to different equivalent classes in $\overline{\mathbb{Q}}$ for different a. We shall denote the equivalence class of a Cauchy sequence $\{a_n\}$ by (a_n), so (a_n) is an element of $\overline{\mathbb{Q}}$. We will think of $\mathbb{Q}$ as a subset of $\overline{\mathbb{Q}}$, identifying $a \in \mathbb{Q}$ with $(\bar{a}) \in \overline{\mathbb{Q}}$. In this notaions, $P = (\bar{0})$.

11. Prove that if $\{a_n\} \sim \{a'_n\}$ and $\{b_n\} \sim \{b'_n\}$ are two pairs of equivalent Cauchy sequences, then $\{a_n \pm b_n\} \sim \{a'_n \pm b'_n\}$ and $\{a_n \cdot b_n\} \sim \{a'_n \cdot b'_n\}$.

12. Let A be an equivalence class in $\overline{\mathbb{Q}}$ different from the class of null sequences P, and let $\{a_n\}$ be any Cauchy sequence in A. Describe the equivalence class A^{-1} such that $AA^{-1} = (1)$.

Thus we showed that $\overline{\mathbb{Q}}$ is a field.

Now we want to define a norm on $\overline{\mathbb{Q}}$ which would extend the norm $|\cdot|$ on $\mathbb{Q}$. Recall that for $x \in \mathbb{Q}$

$$|x| = \begin{cases} x & \text{if } \ x > 0 \\ 0 & \text{if } \ x = 0 \\ -x & \text{if } \ x < 0. \end{cases}$$

Therefore we need to define *positive* and *negative* classes of Cauchy sequences in $\overline{\mathbb{Q}}$. We say that $A > 0$ if it has a representative Cauchy sequence $\{a_n\}$ with all $a_n > 0$. Similarly, $A < 0$ means that it has a representative $\{a_n\}$ with all $a_n < 0$. Notice that if $A > 0$, then $-A < 0$, and vice versa.

Now for any $A \in \overline{\mathbb{Q}}$ we define

$$|A| = \begin{cases} A & \text{if } \ A > 0 \\ 0 & \text{if } \ A = P = (0) \\ -A & \text{if } \ A < 0. \end{cases}$$

Obviously, if $A = \overline{a}$, $a \in \mathbb{Q}$, then $|A| = |a|$.

13. Show that $|\cdot|$ is the norm on $\overline{\mathbb{Q}}$, i.e. that it saftsfies the properties of Execrcise 3.

Thus we extended the norm from $\mathbb{Q}$ to $\overline{\mathbb{Q}}$, and we can talk about bounded, null and Cauchy sequences in $(\overline{\mathbb{Q}}, |\cdot|)$.

14. Prove that $\mathbb{Q}$ is a dense subset of $\overline{\mathbb{Q}}$, i.e. for any $A \in \overline{\mathbb{Q}}$ there exists a sequence of rational numbers $\{a_m\}$ such that $A = \lim_{m\to\infty}(a_m)$ in $(\overline{\mathbb{Q}}, |\cdot|)$.

Now we can prove that $(\overline{\mathbb{Q}}, |\cdot|)$ is complete.

15. Use Exercise 14 to show that every Cauchy sequence in $(\overline{\mathbb{Q}}, |\cdot|)$ converges to an element of $(\overline{\mathbb{Q}}, |\cdot|)$.

Thus we have constructed a normed field which is a completion of $\mathbb{Q}$ with respect to the absolute value norm $|\cdot|$. It is called the field of real numbers and denoted by $\mathbb{R}$.

Notice that the extended norm takes a larger set of values on $\mathbb{R}$ than it did on $\mathbb{Q}$: $\{0\} \cup \mathbb{R}^+$, the set of nonnegative real numbers. We have an *order* on $\mathbb{R}$: $x > y$ iff $x - y > 0$, i.e. $x - y \in \mathbb{R}^+$.

16. Prove that the norm $|\cdot|$ on $\mathbb{R}$ satisfies the *Archimedian property*: given $X, y \in \mathbb{R}$, $x \neq (\bar{0})$, there exists a positive integer n such that $|nx| > |y|$.

17. For which α $|\cdot|^\alpha$ is a norm on $\mathbb{Q}$?

We say that two norms on $\mathbb{Q}$ are *equivalent* if any Cauchy sequence in one norm is a Cauchy sequence in the other, and vise versa.

18. Prove that if $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent (we write $\|\cdot\|_1 \sim \|\cdot\|_2$), then $\|x\|_1 < 1$ iff $\|x\|_2 < 1$, $\|x\|_1 > 1$ iff $\|x\|_2 > 1$, and $\|x\|_1 = 1$ iff $\|x\|_2 = 1$.

19. Use Exercise 18 to show that $|\cdot|^\alpha$, for such α that it is a norm on $\mathbb{Q}$ (cf Exrecise 17), is equivalent to $|\cdot|$.