CONTINUED FRACTIONS

Let $a_0, a_1, \ldots, a_n, \ldots$ be an infinite sequence of positive integers. We define two sequences of integers $\{p_n\}$ and $\{q_n\}$, $n \geq -2$, inductively:

$$\begin{aligned} &p_{-2} = 0,\ p_{-1} = 1;\ p_i = a_i p_{i-1} + p_{i-2} \text{ for } i \geq 0, \\ &q_{-2} = 1,\ q_{-1} = 0;\ q_i = a_i q_{i-1} + q_{i-2} \text{ for } i \geq 0 \end{aligned} \tag{0.1}$$

1. Prove that $1 = q_0 \leq q_1 < q_2 < \cdots < q_n < \ldots$.

Now we define

$$\langle a_0, a_1, \ldots, a_{n-1}, x\rangle = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{x}}}}$$

2. Prove that for any $x \geq 1$ we have

$$\langle a_0, a_1, \ldots, a_{n-1}, x\rangle = \frac{x p_{n-1} + p_{n-2}}{x q_{n-1} + q_{n-2}}\,.$$

3. Prove that $p_{i-1}q_i - p_i q_{i-1} = (-1)^i$ for $i \geq 1$.

Let $r_n = \langle a_0, \ldots, a_n\rangle$. Using Problem 2, we obtain

$$r_n = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} = \frac{p_n}{q_n}\,. \tag{0.2}$$

4. Show that $\{r_n\}$ is a sequence such that

$$r_0 < r_2 < r_4 < \cdots < r_5 < r_3 < r_1,$$

that the limit $\lim_{n\to\infty} r_n$ exists, and this limit is a real number.

Conversely, let α be any real number. We define a sequence of integers $\{a_i\}$, $i = 0, 1, 2, \ldots$ and a sequence of real numbers $\{\alpha_i\}$, $i = 1, 2 \ldots$, inductively:

$$a_0 = [\alpha],\ \alpha_1 = \frac{1}{\alpha - a_0}\quad a_n = [\alpha_n],\ \alpha_{n+1} = \frac{1}{\alpha_n - a_n}\,.$$

This process will terminate if α_n is an integer for some n.

5. Prove that $a_i \geq 1$ for those $i \geq 1$ for which a_i is defined.

Now we can define a sequence $r_n = p_n/q_n$ as in (0.2), which can be infinite or finite, as explained above.

6. Prove that if the sequence $\{r_n\}$ is infinite, then $\lim_{n\to\infty} r_n = \alpha$; if the sequence is finite, $\{r_0, r_1, \ldots, r_n\}$, then $\alpha = r_n$.

Moreover, we deduce that the *convergents* $r_n = p_n/q_n$ are the best approximations of α:

7. If α is irrational, then

$$\left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^2}\,.$$

Thus there is a one–to–one correspondence between the set of real numbers α and the set of sequences (finite and infinite) $a_0, a_1, \ldots, a_n, \ldots$ of integers with

$a_0 \in \mathbb{Z}$ and $a_i \geq 1$ whenever $i \geq 1$. Using this correspondence, prove the following statements.

8. The real number α is rational if and only if the algorithm described above terminates, i.e., α_n is an integer for some n.

Problem 8 can be solved using the Euclidean algorithm.

9. The real number α is a quadratic irrationality, i.e., a real root of the equation $ax^2 + bx + c = 0$ with coefficients $a, b, c \in \mathbb{Z}$, $c \neq 0$, that has two distinct real roots, if and only if its continued fraction expansion is eventually periodic.