LECTURE NOTES FOR THE COURSE "LOCAL RIGIDITY OF GROUP ACTIONS "-PART II BEDLEWO, JULY 2008

1. DIFFEOMORPHISMS OF $\mathbb{T}$

$Diff_+^r(\mathbb{T})$ group of orientation preserving diffeomorphisms of $\mathbb{T}$ of class C^r. $D^r(\mathbb{T})$ is the group of lifts of elements of $Diff_+^r(\mathbb{T})$. $D^r(\mathbb{T}) = \{f = Id + \phi : \phi \in C^r(\mathbb{T}, \mathbb{R}), \phi$ is $\mathbb{Z}$-periodic$\}$. Whenever there is no danger of confusion, we will denote both diffeomorphisms and their lifts by the same notation. Otherwise, f will denote lift of f_T.

For $\alpha \in \mathbb{R}$ by R_α we denote the rotation by α: $R_\alpha(x) = x + \alpha \mod 1$.

1.1. **Rotation number.** For $f \in D^0(\mathbb{T})$ then as $n \to \infty$, $\frac{1}{n}(f^n(x) - x)$ converges uniformly in x to a constant $\rho(f)$. This is *rotation number of f*.

Proposition 1. *$\rho(f)$ has the following properties:*

1. *Let $f = id + \phi$. Then $\rho(\phi) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \phi(f^k)$*
2. *If μ is a probability measure on $\mathbb{T}$ invariant under f, then $\rho(f) = \int_{\mathbb{T}} \phi(x)d\mu$.*
3. *$\rho(R_p \circ f) = p + \rho(f)$ for $p \in \mathbb{Z}$, thus ρ factors to a map $\rho : Diff_+^0(\mathbb{T}) \to \mathbb{T}$*
4. *If $f, g \in D^0(\mathbb{T})$ and h is a homeomorphism which conjugates f and g then $\rho(f) = \rho(g)$.*
5. *$\rho(f)$ is irrational iff f (as a diffeomorphism of $\mathbb{T}$) has no periodic orbits.*
6. *$\rho(f)$ is irrational than f on $\mathbb{T}$ is uniquely ergodic.*
7. *If $f \circ g = g \circ f$ then $\rho(f \circ g) = \rho(f) + \rho(g)$.*

Proof: (7) Let $f = Id + \phi$ and $g = Id + \psi$. If μ is a probability measure invariant both by f_T and g_T (which exists by Markov-Kakutani Theorem), then $f \circ g = Id + \psi + \phi \circ g$. Thus $\rho(f \circ g) = \mu(\psi + \phi \circ g) = \mu(\psi) + \mu(\phi \circ g) = \mu(\psi) + \mu(phi) = \rho(f) + \rho(g)$ $\square$
The question of conjugating a diffeomorphism of $\mathbb{T}$ with irrational rotation number to a rotation on $\mathbb{T}$ was first raised by Poincare.

Theorem 1 (Denjoy). *A C^2 diffeo of $\mathbb{T}$ with no periodic points is topologically conjugated to irrational rotation.*

Denjoy counterexample: For irrational α for every $\epsilon > 0$ there exists $f \in D^{2-\epsilon}(\mathbb{T})$ such that there is no topological conjugacy to R_α.

Remark 1. (1)

Question 1 (Arnold). *: What is the smoothness of the conjugacy in the Denjoy theorem, depending on the smoothness of f and $\rho(f)$? In other words this is asking: is the statistical distribution of f orbits in $\mathbb{T}$ (the invariant measure) given by smooth density? C^∞-density?*

In the subsequent sections we will state the answers to the above question of Arnold.

Definition 1. $O_\alpha^r = \{h^{-1} \circ R_\alpha \circ h : h \in D^r(\mathbb{T})\}$,
$F_\alpha^r = \{f \in D^r(\mathbb{T}) : \rho(f) = \alpha\}$. By (4) of Proposition 1.1 it is clear that $O_\alpha^r \subset F_\alpha^r$.

The question of Arnold can be interpreted as:
(Local). Given α irrational, determine k, r, l and condition on α, such that there a neighborhood U of R_α in some C^k topology such that $U \cap F_\alpha^r \subset O_\alpha^l$.
(Global) Given α irrational, determine r, l and condition on α, such that $F_\alpha^r \subset O_\alpha^l$.
Note that Denjoy theorem gives: $F_\alpha^k \subset O_\alpha^0$ for any $k \geq 2$.

1.2. **Local result for diffeomorphisms of $\mathbb{T}$ with Diophantine rotation number.**

1.2.1. *Set-up of the problem, heuristics.* We wish to study small perturbations of rotations. Therefore we look at $f = R_\alpha + \tilde{f} \in D^\infty(\mathbb{T})$ where $\alpha \in \mathbb{R}$ and $f \sim R_\alpha$ in some C^k topology, i.e. assume that $\|\tilde{f}\|_{C^r} \sim \epsilon << 1$. If $\rho(f) = \alpha$ is irrational, from Denjoy theorem there exists a $h = id + \tilde{h}$ which gives a homeomorphism on $\mathbb{T}$, but the theorem produces no smoothness for h. So the idea of Arnold, Moser and Kolmogorov was to construct this conjugacy anew, and to obtain smoothness from the construction.

The conjugacy we look for $h = id + \tilde{h}$ should satisfy $\tilde{h} \sim \epsilon$ and

$$R_\alpha \circ h = h \circ f$$

That is:

$$\tilde{h}(R_\alpha + \tilde{f}) - \tilde{h} = -\tilde{f} \tag{1.1}$$

The idea for solving (1.1) comes from the Newton method: first linearize the non-linear problem at the point which represents our initial guess for the solution, resolve the linear problem, use the solution to linear problem to construct a new approximate solution to the non-linear problem which is much closer to being an actual solution than our initial guess. If this can be done so that this process converges fast enough and in nice enough spaces so that the limit exists, then in the limit one gets an exact solution to the non-linear problem.

The linearized equation for our non-linear problem (1.1) is:

$$\tilde{h}(R_\alpha) - \tilde{h} = -\tilde{f} \tag{1.2}$$

We will denote the operator which takes $\tilde{h}$ to $\tilde{h}(R_\alpha) - \tilde{h}$ by L. SO the question is whether this operator can be inverted, and in which space this can be done.

First question to ask is what are the obstructions to inverting L and there is one obvious obstruction: the average of $\tilde{f}$:

(Obstructions-1): $[\tilde{f}] = 0$.

Now assume that this obstruction vanishes, then can one solve the linearized problem?

Passing to Fourier coefficients, for every $n \neq 0$: $\tilde{h}(n) = \frac{\tilde{f}(n)}{1-e^{2\pi i n\alpha}}$. If we want $\tilde{h}$ to be C^∞ we need that its Fourier coefficients decay faster than any polynomial. Since $\tilde{f}$ is C^∞ then $\tilde{f}(n)$ decay faster than any polynomial, so in order to claim the same for $\tilde{h}(n)$ we need an upper polynomial bound for $\frac{1}{|1-e^{2\pi i n\alpha}|}$, something like:

$$\frac{1}{|1-e^{2\pi i n\alpha}|} < |n|^\tau$$

for some $\tau > 0$. Numbers α which satisfy this condition for all $n \in \mathbb{Z} \setminus \{0\}$ are Diophantine numbers with exponent $\tau > 0$.

1.2.2. *Diophantine condition.* Let $\tau > 0$ and $C > 0$. For $\alpha \in \mathbb{R}$, let $\|n\alpha\| = \min_{p\in\mathbb{Z}} |n\alpha - p|$. Then:

$$DC(\tau, C) = \{\alpha \in \mathbb{R} \setminus \mathbb{Q} : \forall n \in \mathbb{Z} \setminus \{0\}, \|n\alpha\| \geq \frac{C}{|n|^\tau} \tag{1.3}$$

We will use the following notation: $DC(\tau) = \cup DC(\tau, C)$, $DC = \cup D(\tau)$.

Some properties of the sets of numbers satisfying Diophantine condition:

- $DC(\tau, C) \subset DC(\tau', C')$ for $\tau \leq \tau'$ and $C \geq C'$.

- $DC(\tau, C)$ is closed and has empty interior. Thus DC and $DC(\tau)$ are small in the topological sence (Baire category).
- For $\tau > 0$, $Leb(DC(\tau)) = 1$.
- Any irrational algebraic number is in $DC(\tau)$ for all $\tau > 0$.
- Any irrational algebraic number of degree 2 has eventually periodic continued fractions expansion, and vice versa. Such numbers are in $DC(0)$. It is not known (??) whether algbraic irrationals of degree greater than 3 are in $D(0)$

1.2.3. *Coboundary equation.* Some notation:
-We will use notation $\|\cdot\|_r$ for $\|\cdot\|_{C^r}$.
-We will use notation $|\cdot|_r$ for $|f|_r = sup|f(n)||n|^r$.
-Relations between the two norms are: $\|f\|_r \leq \|f\|_{r+2}$, $|f|_r \leq \|f\|_r$

Lemma 1. *Let $\alpha \in DC(\tau, C)$, let $\tilde{f} \in C^\infty(\mathbb{T}, \mathbb{R})$ and let $[\tilde{f}]$ denote the average of f. Then there exists a solution $\tilde{h}$ to the equation*

$$\tilde{h} \circ R_\alpha - \tilde{h} = \tilde{f} - [\tilde{f}] \tag{1.4}$$

such that

$$\|\tilde{h}\|_r \leq C_r \|\tilde{f}\|_{r+\sigma} \tag{1.5}$$

for all $r \geq 0$ and $\sigma > 2 + \tau$.

Note: If α is not Diophantine then there exists $\tilde{f} \in D^\infty(\mathbb{T})$ such that the solution $\tilde{h}$ is not even a distribution. Namely, for α Liouville, there exists a sequence n_k of integers such that $|e^{2\pi i n_k \alpha} - 1| \leq C|n_k|^{-\tau}$. Then define $\tilde{f}$ by choosing its only non-zero Fourier coefficients to be $\tilde{f}(n_k) = |e^{2\pi i n_k \alpha} - 1|^{1/2}$. Compute $\tilde{h}$ for this $\tilde{f}$ and check that its Fourier coefficients do not even decay.

Proof: For $n \neq 0$, $\tilde{h}(n) = \frac{\tilde{f}(n)}{|e^{2\pi i n_k \alpha} - 1|}$. Since $C_1\|n\alpha\| \leq |e^{2\pi i n_k \alpha} - 1| \leq C_2\|n\alpha\|$ and α is Diophantine it follows that $|e^{2\pi i n_k \alpha} - 1|^{-1} \leq C|n|^\tau$. Thus $|\tilde{h}(n)| \leq C|\tilde{f}(n)||n|^\tau$. So we have:

$$\|\tilde{h}\|_{C^r} \leq C_r|\tilde{h}|_{r+2} = C_r \sup_n |\tilde{h}(n)||n|^{r+2} \leq C_r \sup_n |\tilde{f}(n)||n|^{r+\tau+2} \leq C_r\|\tilde{f}\|_{C^{r+\sigma}}$$

□

It is clear from the above Lemma that there is certain *loss of regularity* for the solution, in other words the operator $L : \tilde{h} \mapsto \tilde{h} \circ R_\alpha - \tilde{h}$ takes C^r maps to C^r maps, but its inverse takes C^r maps to only $C^{r-\sigma}$ maps. This is clearly a problem if our goal here is to repeat this process of lineariation *ad infinitum*.

Standard method which allows us to overcome this obstacle was used by Moser, and subsequently in all similar and/or more general situations (see Section **??**), is to use a family of smoothing operators S_t which act on our function spaces $C^r(\mathbb{T}, \mathbb{R})$. The idea is to solve the linear problem i.e. the coboundary equation in Lemma 1 not for $\tilde{f}$ but for $S_t\tilde{f}$, map which takes $\tilde{f}$ from $C^r(\mathbb{T}, \mathbb{R})$ to $C^{r+t}(\mathbb{T}, \mathbb{R})$, so that the solution of the coboundary problem is again in $C^r(\mathbb{T}, \mathbb{R})$. This is not quite how it is done because the residue $\tilde{f} - S_t\tilde{f}$ should be made very small with respect to $\tilde{f}$, but this is the general idea.

Proposition 2. *Let X be a compact space, then for the space of C^∞ functions on X there exists a collection of smoothing operators $S_t : C^\infty(X) \to C^\infty(X)$, $t > 0$, such that the following holds:*

$$\begin{aligned} \|S_t f\|_{s+s'} &\leq C_{s,s'} t^{s'} \|f\|_s \\ \|(I - S_t) f\|_{s-s'} &\leq C_{s,s'} t^{-s'} \|f\|_s \end{aligned} \tag{1.6}$$

For the construction of smoothing operators see: Example 1.1.2. (2), Definition 1.3.2, Theorem 1.3.6, Corollary 1.4.2 in **[?]**

Note: we may assume that $[S_t f] = [f]$.

The following is a classical fact, but also easy to prove from the existence of smoothing family.

Corollary 1. *In the set-up of the previous proposition, the following interpolation inequalities hold in $C^\infty(X)$: $\|f\|_r \leq C_{\lambda,r_1,r_2} \|f\|_{r_1}^{1-\lambda} \|f\|_{r_2}^{\lambda}$ where $r = (1-\lambda)r_1 + \lambda r_2$.*

By combining smoothing operators and Lemma 1 we obtain:

Lemma 2. *Let $\alpha \in DC(\tau, C)$, let $\tilde{f} \in C^\infty(\mathbb{T}, \mathbb{R})$ and let $[\tilde{f}]$ denote the average of f. Then there exists $\tilde{h}$ which solves the equation*

$$L(\tilde{h}) = \tilde{h} \circ R_\alpha - \tilde{h} = S_t\tilde{f} - [\tilde{f}] \tag{1.7}$$

such that

$$\|\tilde{h}\|_r \leq C_r t^\sigma \|\tilde{f}\|_r \tag{1.8}$$

for all $r \geq 0$ and $\sigma > 2 + \tau$.

Remark 2. Note that for any $k \leq r$ by the properties of the smootthing operators, the following estimate holds as well: $\|\tilde{h}^{(n)}\|_r \leq C_r t_n^{\sigma+k} \|\tilde{f}^{(n)}\|_{r-k}$

1.2.4. *Statement of the local result and proof.*

Theorem 2 (Arnold, Moser). *(1) Let $\tau \geq 0$, $\alpha \in DC(\tau)$, $r_0 > 2+\tau$. There exists a neighborhood U of R_α in $D^{r_0}(\mathbb{T})$ such that if $r \geq r_0$ satisfies $r \notin \mathcal{N}$, $s = r-1-\tau \notin \mathcal{N}$, then $F^r_\alpha(\mathbb{T}) \cap U \subset O^s_\alpha$.*

(2) In particular: There exists $r_0 > 0$ and a neighborhood U of R_α in D^{r_0} such that $F^\infty_\alpha(\mathbb{T}) \cap U \subset O^\infty_\alpha$.

Remark 3. In case both numbers in the theorem are in $\mathbb{N}$, the condition $r-s = 1+\tau$ is replaced by $r-s > 1+\tau$.

Remark 4. In fact, the arithmetical conditions are optimal (Herman?): Let $r \geq s \geq 1, \tau = r-s-1, \alpha \notin DC(\tau)$. Then there exists arbitrary near R_α in the C^r topology, diffeomorphisms $f \in D^r(\mathbb{T})$ with $\rho(f) = \alpha$ which are not in O^s_α.

Remark 5. It is worth noting here that this rigidity phenomenon which appears in Theorem 2 is considered to be quite rare in rank-one situation. The following is a question asked by Herman: **Question**[Herman] Let M be a compact C^∞ manifold, $f \in Diff^\infty(M)$. Let $\mathcal{U}$ be a small C^∞-neighborhood of Id. Let $O_{f,\mathcal{U}} := \{g \circ f \circ g^{-1} : g \in \mathcal{U}\}$. If $O_{f,\mathcal{U}}$ is a finite codimension manifold, is it true that $M = \mathbb{T}^N$ and f is smoothly conjugate to a Diophantine rotation on $\mathbb{T}^n$? (*Note*. Compare this with the Greenfield-Wallach-Katok conjecture stated in Flaminio-Forni lectures).

In what follows we will prove the statement (2) of Theorem 2.

Lemma 3. *Assume that $f = R_\alpha + \tilde{f}$ is in $D^\infty(\mathbb{T})$, with $\|\tilde{f}\|_1 < 1$. Assume that there exists a $\tilde{h} \in C^\infty(\mathbb{T})$ such that:*
(i) $\|L\tilde{h} - (S_t\tilde{f} - [\tilde{f}])\|_0 \leq Ct^\sigma \|\tilde{f}\|_0 \|\tilde{f}\|_1$
(ii) $\|\tilde{h}\|_r \leq Ct^\sigma \|\tilde{f}\|_r$, $r \geq 0$
(iii) $\|\tilde{h}\| < \frac{1}{4}$ and $h = id + \tilde{h}$, h^{-1} exists.

Then for the map $\tilde{f}^{(1)} := h^{-1} \circ f \circ h - R_\alpha$ we have:
(1) $\|\tilde{f}^{(1)}\|_0 \leq Ct^\sigma \|\tilde{f}\|_0 \|\tilde{f}\|_1 + C_l t^{-l} \|\tilde{f}\|_l$, for all $l \geq 0$
(2) $\|\tilde{f}^{(1)}\|_l \leq C_l t^\sigma (1 + \|\tilde{f}\|_l)$, for all $l \geq 0$.

Remark 6. By comparing Lemma 2 and condition (i) in Lemma above, it is obvious that instead of the exact solution in Lemma 2 it would suffice to have a solution with certain error, namely the existence of $\tilde{h}$ satisfying $\|L\tilde{h} - (S_t\tilde{f} - [\tilde{f}])\|_0 \leq Ct^\sigma \|\tilde{f}\|_0 \|\tilde{f}\|_1$ is enough for the application of Lemma 3. This observation will be crucial for the application of this method to higher rank actions in subsequent sections.

Proof: Let $\tilde{f}^{(1)} := h^{-1} \circ f \circ h - R_\alpha = f^{(1)} - R_\alpha$. Then since $h = id + \tilde{h}$ and $f = R_\alpha + \tilde{f}$ we have:

$$
\begin{aligned}
h \circ \tilde{f}^{(1)} &= f \circ h \\
\tilde{f}^{(1)} &= \tilde{h} - \tilde{h} \circ f^{(1)} + \tilde{f} \circ h \\
&= \tilde{h} - \tilde{h} \circ R_\alpha + \tilde{h} \circ R_\alpha - \tilde{h} \circ f^{(1)} + \tilde{f} \circ h \\
&= -L\tilde{h} + \tilde{h} \circ R_\alpha - \tilde{h} \circ f^{(1)} + \tilde{f} \circ h \\
&= -(L\tilde{h} - S_t\tilde{f} + [\tilde{f}]) + (\tilde{f} \circ (Id + \tilde{h}) - \tilde{f}) + [\tilde{f}] + \\
&\quad (\tilde{h} \circ R_\alpha - \tilde{h} \circ f^{(1)}) + (\tilde{f} - S_t\tilde{f})
\end{aligned} \tag{1.9}
$$

Enumerate parentheses in the expression in the last line above from left to right by (1), (2), (3), (4) and (5). We will estimate each term in order to obtain C^0 estimate for $\tilde{f}^{(1)}$.

(1) By assumption (i), $\|L\tilde{h} - (S_t\tilde{f} - [\tilde{f}])\|_0 \leq Ct^\sigma \|\tilde{f}\|_0 \|\tilde{f}\|_1$.

(2) $\|\tilde{f} \circ (Id + \tilde{h}) - \tilde{f}\|_0 \leq C\|\tilde{f}\|_1 \|\tilde{h}\|_0 \leq Ct^\sigma \|\tilde{f}\|_1 \|\tilde{f}\|_0\|$, by using assumption (ii) for $r = 0$.

(3) $|[\tilde{f}]|$ can be absorbed by the left hand side, i.e. by the C^0 norm of $\tilde{f}^{(1)}$. The reason tor this lies in the assumption that $\rho(f) = \alpha$. From this it follows that $\rho(f^{(1)}) = \alpha$ since the rotation number is conjugacy invariant. Now since $f^{(1)} = R_\alpha + \tilde{f}^{(1)}$ and its rotation number is α, it follows that $\tilde{f}^{(1)}$ has a zero. Then for any constant C we have: $\|f\|_0 \leq \max_{f(x) \geq 0} f(x) + \max_{f(x)<0}(-f(x)) \leq \max_{f(x) \geq 0}(f(x) - C) + \max_{f(x)<0}(-(f(x) - C)) \leq 2\max_x |f(x) - C| = 2\|f - C\|_0$. In place of constant C we can put $[\tilde{f}]$ to obtain $\|f\|_0 \leq 2\|f - [\tilde{f}]\|_0$. *Note*: another approach to take care of the average is to carry out the iteration with average accumulating at some point so that in the limit one gets that the inital perturbation needs to be adjusted by a certain rotation in order to have a conjugacy to a rotation, and then the rotation number assumption implies that this rotation adjustment has to be trivial.

(4) This term can also be absorbed by the left hand side due to the assumption (iii) on smallness of C^1 norm of $\tilde{h}$. Namely:

$$\|\tilde{h} \circ R_\alpha - \tilde{h} \circ f^{(1)}\|_0 \leq \|\tilde{h}\|_1 \|\tilde{f}^{(1)}\|_0 \leq \frac{1}{4}\|\tilde{f}^{(1)}\|_0$$

(5) The last term is estimated by using properties of smoothing operators:

$$\|\tilde{f} - S_t\tilde{f}\|_0 \leq Ct^{-l}\|\tilde{f}\|_l$$

From (1)-(5) the first estimate follows.

The second is a consequence of the general fact about conjugacy equation that from $f^{(1)} = h^{-1} \circ f \circ h$ there follows a 'linear' estimate in any C^r norm: $\|f^{(1)}\|_l \leq C_l(1 + \|h\|_l + \|f\|_l)$. (this $\|\tilde{f}\|_1 < 1$). Combining this with the assumption (ii) we have $\|f^{(1)}\|_l \leq C_l t^{\sigma}(1 + \|\tilde{f}\|_l)$.

(*Note*. This last fact is in fact very general and very useful in this set-up as was pointed out by Zehnder. Namely, any operator $F(x, D^n f, D^m h)$ on spaces of functions which involves only partial derivatives or functional substitution grows at most linearly with $|f|_{s+n}|h|_{s+m}$. For example $\|f \circ h\|_s \leq C(|f|_s + |f|_1|h|_s$.)

Note. It may be easier to prove all this inequalities for the norm defined as $|||v|||_r := \sup_x |\frac{d^r v}{dx^r}|$, instead of $\|\cdot\|$.

□

Proof of Theorem 2

Now we use Lemmas 2 and 3 to construct a sequence of perturbations f^n and a sequence of conjugacy maps h^n which converge.

To begin this process, let $f^{(0)} = f$, our initial perturbation. Let $h^{(0)} = id$. For $n \geq 1$, given $f^{(n)} = R_\alpha + \tilde{f}^{(n)}$, apply Lemma 2 with specific t_n, to $\tilde{f}^{(n)}$ to obtain $\tilde{h}^{(n)}$ and define $h^{(n)} = Id + \tilde{h}^{(n)}$. Assuming that invertibility of $h^{(n)}$ is assured, define

$$f^{(n+1)} = (h^{(n)})^{-1} \circ f^{(n)} \circ h^{(n)}$$

Set:

speed of convergence $\kappa = 4/3$
C^0-error at n-th step of iteration $\epsilon_n := \epsilon^{(\kappa^n)}$
choice of smoothing operator at n-th step $t_n = \epsilon^{-\frac{1}{3(\sigma+1)}}$
The upper norm which is controlled $l = 8(\sigma + 1), \sigma = \tau + 2$

Claim. Under the assumptions, the estimates:

$$(1.10) \qquad \begin{aligned} &(1)\ \|\tilde{f}^{(n)}\|_0 < \epsilon_n \\ &(2)\ \|\tilde{f}^{(n)}\|_l < \epsilon_n^{-1} \\ &(3)\ \|\tilde{h}^{(n)}\|_1 < \epsilon_n^{1/2} \end{aligned}$$

hold for all $n \in$.

Proof of the Claim Is by induction. Let us assume that we have constructed $f^{(n)}$ and $h^{(n)}$ for which the statements of the Claim hold.

From Lemma 2 we have the following estimates:

$$L(\tilde{h}^{(n)}) = S_{t_n}\tilde{f}^{(n)} - [\tilde{f}^{(n)}] \tag{1.11}$$

and

$$\|\tilde{h}^{(n)}\|_r \le C_r t_n^{\sigma+k}\|\tilde{f}^{(n)}\|_{r-k} \tag{1.12}$$

(see the remark after Lemma LEsmoothing) Since these estimates satisfy the conditions in Lemma 3, we have for $\tilde{f}^{(n+1)}$:

$$\|\tilde{f}^{(n+1)}\|_0 \le Ct_n^{\sigma}\|\tilde{f}^{(n)}\|_0\|\tilde{f}^{(n)}\|_1 + C_l t_n^{-l}\|\tilde{f}^{(n)}\|_l \tag{1.13}$$

$$\|\tilde{f}^{(n+1)}\|_l \le C_l t_n^{\sigma}(1+\|\tilde{f}^{(n)}\|_l) \tag{1.14}$$

From (1.14) it follows that

$$\|\tilde{f}^{(n+1)}\|_l \le 2C_l t_n^{\sigma}\epsilon_n^{-1} = 2C_l\epsilon_n^{-\frac{1}{3}\frac{\sigma}{\sigma+1}}\epsilon_n^{-1} < \epsilon^{-\frac{1}{3}-1} = \epsilon_{n+1}^{-1} \tag{1.15}$$

This proves inequality (2) of the Claim. Similarly, the inequality (3) of the Claim follows from (1.12) with $k=1$:

$$\|\tilde{h}^{(n)}\|_1 \le Ct_n^{\sigma+1}|\tilde{f}^{(n)}\|_0 < Ct_n^{\sigma+1}\epsilon_n = C\epsilon_n^{1-\frac{1}{3}} < \epsilon_n^{1/2} \tag{1.16}$$

Finally, check that (1) holds for $f^{(n+1)}$ if it holds for $f^{(n)}$. Here we will see where l comes from. How big l we need to take depends on the speed of convergence and on σ. From (1.13):

$$\|\tilde{f}^{(n+1)}\|_0 \le C(t_n^{\sigma}\epsilon_n^{2(1-\frac{1}{l}} + t_n^{-l}\epsilon_n) \le 2C\epsilon_n^x + \epsilon_n^y \tag{1.17}$$

where $x = -\frac{1}{3}\frac{\sigma}{\sigma+1} + 2(1-\frac{1}{l})$ and $y = \frac{l}{3(\sigma+1)}$ In order to have $\|\tilde{f}^{(n+1)}\|_0 \le \epsilon_{n+1} = \epsilon_n^{4/3}$, we need $x > \frac{4}{3}$ and $y > \frac{4}{3}$. Both of these conditions are satisfied when $l \ge 8(\sigma+1)$, and we chose the smallest such $l = 8(\sigma+1)$.

This concludes the proof of the Claim which implies existence of a C^1 conjugacy $\mathcal{H}$ in the limit of the sequence $\mathcal{H}_n := h^{(n)} \circ h^{(n-1)} \circ h^{(n-2)} \circ \cdots \circ h^{(0)}$ providing ϵ is small enough.

To prove convergence in any C^k suggestion of Zehnder was to use interopolation inequalities. From (1.14):

$$\begin{aligned}\|\tilde{f}^{(n)}\|_m &\le C_m t_{n-1}^{\sigma}(1+\|\tilde{f}^{(n-1)}\|_l) \le \epsilon_{n-1}^{-\frac{1}{3}}(1+\|\tilde{f}^{(n-1)}\|_l)\\ &\le C_m\prod_{i=1}^{n-1}\epsilon_i^{-\frac{1}{3}}(1+\|\tilde{f}\|_m) \le D_m\epsilon_n^{-1}\end{aligned}$$

where in the second line above the constant D_m is $D_m := C_m(1 + \|\tilde{f}\|_m)$. Naw take $m = 3k$. Then:

$$\|\tilde{f}^{(n)}\|_k \le C_k \|\tilde{f}^{(n)}\|_{C^0}^{\frac{2}{3}} \|\tilde{f}^{(n)}\|_{3k}^{\frac{1}{3}} < C_k \epsilon_n^{\frac{2}{3}} \epsilon_n^{-\frac{1}{3}} = C_k \epsilon_n^{\frac{1}{3}}$$

$$\|\tilde{h}^{(n)}\|_k \le C_k t_n^{\sigma} \|\tilde{f}^{(n)}\|_k \le C_k \epsilon_n^{-\frac{\sigma}{3(\sigma+1))}} \epsilon_n^{\frac{1}{3}} = C_k \epsilon_n^{\delta}$$

with $\delta = \dfrac{1}{3(\sigma+1)} > 0$ and the constant C_k changing throughout the above procedure, but depending only on k and the C^{3k} norm of the initial perturbation $\tilde{f}$.

This implies the convergence of the sequence $\mathcal{H}_n$ in C^k norm for every $k \in \mathbb{N}$ i.e. the limit $\mathcal{H}$ is a C^∞ map.

1.3. **Global result for diffeomorphisms of $\mathbb{T}$ with Diophantine rotation number.**

Theorem 3 (Herman, Yoccoz)**.** *If $\alpha \in D(C, \tau)$ for some $C > 0, \tau > 0$, $F_\alpha^\infty \subset O_\alpha^\infty$.*

1.4. $\mathbb{Z}^k$ **actions on $\mathbb{T}$ and simultaneously Diophantine numbers.**

Let f, g be smooth commuting diffeomorphisms. Then they generate a $\mathbb{Z}^2$ action α on $\mathbb{T}$ by $\alpha((n,m),x) = f^n g^m(x)$. We assume that all ρ are irrational.

Since the rotation number function is a homomorphisms on commutative subgroups in $D^\infty(\mathbb{T})$, we have that $\rho(f^n g^m) = n\rho(f) + m\rho(g)$. We assume that $\rho(f^n g^m)$ are irrational.

If one element of the action, say f is topologically conjugated to a rotation $\mathbb{R}_\alpha$ with α irrational, then all other other elements of the action are conjugated to rotations via the same conjugacy. Indeed, if f is conjugate to a rotation, then conjugates of all the other elements commute with that irrational rotation. Since centralizer of R_α consists of rotations, it follows that the whole action is conjugate to action by rotations.

To see that the centralizer of rotations consists of rotations, it is enough to look at $f \circ R_\alpha = R_\alpha \circ f$, for $f = id + \tilde{f}$. Then $\tilde{f} \circ R_\alpha = \tilde{f}$ i.e. $\tilde{f}$ is constant on every orbit, and since $\mathbb{R}_\alpha$ is topologically transitive and $\tilde{f}$ continuous, $\tilde{f}$ is constant everywhere. So f is a rotation.

Thus the Denjoy theorem as in the rank-one case produces a homeomorphism.

We note here that in the Denjoy theorem for commuting diffeomorphisms recently obtained by Navas et al, [**?**], the topology required is $C^{1+\phi}$ where $\phi > 1/d$ and d is the dimension of the action.

To the best of my knowledge, the case $\phi = 1/d$ hasn't been resolved yet.

However, as in the rank-one case, Denjoy theorem gives only a homeomomorphism.

Now if we put more stringent conditions on rotation numbers, namely if we assume that rotation number of at least one element of the action has Diophantine rotation number, then it is possible to use Herman's global result to obtain C^∞ conjugacy to a rotation, and by the above discussion the same conjugacy works for all other elements of the action.

This raised the question of weather Herman's rank-one result is applicable to all higher-rank actions which conjugate smoothly to rotations. In other words the question is:

Question. Do there exist for example $\mathbb{Z}^2$-actions by smooth circle diffeomorphisms which are smoothly conjugate to rotations but all the rotation numbers of action elements are all Liouville?

This question was completely answered by Moser in 1990 [] for the local set-up, and in the global situation it was answered recently by Fayad and Khanin [].

Before we state Moser's results and prove the local one, we will try to apply the same reasoning as in the rank-one case for small perturbations of actions by rotations, and try to come up with necessary conditions for this method to work.

Remark 7. We include here a remark made by Moser that if a group of circle mappings generated by $f_1, f_2, \dots, f_n$ contains only fixed point free elements then this group is necessarily commutative. References produced are due to J. Mather, namely this is a consequence of a theorem on ordered Archimedean groups which goes all the way back to Hölder.

1.4.1. *Set-up and heuristics.* Let $< f, g > \sim < R_\alpha, R_\beta >$ in some C^k topology, $f, g \in D^\infty(\mathbb{T})$ and $f \circ g = g \circ f$. Let $f = R_\alpha + \tilde{f}$ and $g = R_\beta + \tilde{g}$ with $\tilde{f}, \tilde{g} \sim \epsilon << 1$. Let $F^\infty_{(\alpha,\beta)} = \{(f, g) : \rho(f) = \alpha, \rho(g) = \beta\}$.

The conjugacy we look for $h = id + \tilde{h}$ should satisfy $\tilde{h} \sim \epsilon$ and

$$R_\alpha \circ h = h \circ f, \;\; R_\beta \circ h = h \circ g$$

. That is:

$$\begin{aligned} \tilde{h}(R_\alpha + \tilde{f}) - \tilde{h} &= -\tilde{f} \\ \tilde{h}(R_\beta + \tilde{g}) - \tilde{h} &= -\tilde{g} \end{aligned} \tag{1.18}$$

The linearized equation for our non-linear problem (1.18) is:

$$\begin{aligned} \tilde{h}(R_\alpha) - \tilde{h} &= -\tilde{f} \\ \tilde{h}(R_\beta) - \tilde{h} &= -\tilde{g} \end{aligned} \tag{1.19}$$

We will denote the operator which takes $\tilde{h}$ to $\tilde{h}(R_\alpha) - \tilde{h}$ by L_α and the operator which takes $\tilde{h}$ to $\tilde{h}(R_\beta) - \tilde{h}$ by L_β. Also let $L = (L_\alpha, L_\beta)$. As before, the question is whether this operator can be inverted, and in which space this can be done.

First let us look for the obstructions to existence of any kind of a common solution to (1.19). Unlike the rank-one case we had earlier, here the averages are not the only obstructions.

Obstructions-2:
(1) $[\tilde{f}] = [\tilde{g}] = 0$
(2) $L_\alpha \tilde{g} = L_\beta \tilde{f}$

Now we assume that the obstructions above vanish, and we try to produce a solution h by using Fourier expansions as before:

$$\tilde{h}(n) = \frac{\tilde{f}(n)}{e^{2\pi i n\alpha} - 1} = \frac{\tilde{f}(g)}{e^{2\pi i n\beta} - 1}$$

Therefore to estimate $\tilde{h}(n)$ it is enough to have a lower bound for *at least one* of the numbers $|e^{2\pi i n\alpha} - 1|$ and $|e^{2\pi i n\beta} - 1|$. In other words, in ordr to obtain C^∞ h what is needed is that for some $\tau > 0$ and some constant $C > 0$, $\max\{\|n\alpha\|, \|n\beta\|\} \geq C|n|^{-\tau}$. This is known as a *simoultaneous Diophantine condition*.

So it seems like this condition suffices for carrying out the iteration scheme. The obstructions (1) should be taken care of as in the rank-one case if we assume that $\rho(f)$ and $\rho(g)$ satisfy the simultaneous Diophantine condition. However, two non-trivial problems remain, namely that the obstruction (2) does not vanish, and the second one would be to show that it make sense to consider actions generated by f, g with $\rho(f)$ and $\rho(g)$ simult. Diophantine, i.e. that there exist such actions without elements with Diophantine rotation number.

The former problem we discuss later, we first mention the resolution of the latter:

Theorem 4 (Moser). *For $\tau > 2$ (and in fact for $\tau = 2$ as well, which was proved by Masser), there exists a set of cardinality of continuum of vectors $(\alpha, \beta) \in \mathbb{R}^2$ which are simultaneously Diophantine with constants C and τ, but such that all ratios $\frac{j_0 + j_1\alpha + j_2\beta}{l_0 + l_1\alpha + l_2\beta}$ are Liouville where $(j_0, j_1, j_2), (l_1, l_2, l_3) \in \mathbb{Z}^3$*

Remark 8. For $\tau < 1$ there are no such vectors as described in Theorem above. This follows from Khinchine's transference principle. And for $1 \leq \tau < 2$ it is still an open problem whether such vectors exist.

1.4.2. *Local result for $\mathbb{Z}^k$ actions on $\mathbb{T}$.*

Theorem 5. *Let (α, β) be simult. Diophantine with constants C and τ. Then there exist neighborhoods $\mathcal{U}_\alpha$ of R_α and $\mathcal{U}_\beta$ of R_β such that any pair of commuting f and g such that $(f, g) \in \mathcal{U}_\alpha \times \mathcal{U}_\beta \cap F^\infty_{(\alpha,\beta)}$ there exists a C^∞ diffeomorphism which conjugates both f and g to rotations.*

From the above heuristics and the proof of rank-one counterpart, Theorem 2, it is clear that the proof of Theorem 2 goes through here verbatim, modulo Lemma 2. As we remarked after the statement of Theorem 2, it suffices to obtain an approximate solution to the linearized problem instead of an actual one, i.e. it is enough to obtain a solution which would satisfy an estimate (i) in Lemma 3. If this is done, and this approximate solution to linear problem (1.19) satisfies (i), (ii) and (iii) of Lemma 3, then the proof of Theorem 2 applies here as well and gives the conjugacy required here.

The main problem with solving (approximately) the linear problem in rank-two is the existence of additional assumptions. In fact obstruction (2) merely states that $\tilde{f}$ and $\tilde{g}$ generate a cocycle over the linear action $< R_\alpha, R_\beta >$. However, this is only approximately true, with error of order of ϵ^2. Namely from commutativity assumption on f and g we have:

$$\|L_\alpha \tilde{g} - L_\beta \tilde{f}\|_0 \leq \|\tilde{g}\|_1 \|\tilde{f}\|_0 + \|\tilde{g}\|_0 \|\tilde{f}\|_1$$

Main part of the proof is to show that $\tilde{g}$ and $\tilde{f}$ can be approximated by $\tilde{g}_0$ and $\tilde{f}_0$ to the order of ϵ^2 (or so...) so that for $\tilde{g}_0$ and $\tilde{f}_0$ the obstruction (2) vanish, i.e. so that $\tilde{g}_0$ and $\tilde{f}_0$ generate a cocycle over the action $< R_\alpha, R_\beta >$.

Denote by $K : (\tilde{f}, \tilde{g}) \mapsto L_\alpha \tilde{g} - L_\beta \tilde{f}$ then we are trying to split a pair $(\tilde{f}, \tilde{g})$ into one part which is in the image of L (and in the kernel of K) and the other part which is in the preimage of $K(\tilde{f}, \tilde{g})$, and whose size is comparable to that of $K(\tilde{f}, \tilde{g})$. I other words we are trying to find two tame maps $L^* : (\tilde{f}, \tilde{g}) \mapsto L^*(\tilde{f}, \tilde{g}) \in C^\infty(\mathbb{T})$ and $K^* : K(\tilde{f}, \tilde{g}) \mapsto K^* K(\tilde{f}, \tilde{g}) \in C^\infty(\mathbb{T})^2$ in such a way that

$$LL^* + K^*K = I$$

.

Moser resolved this issue by constructing formal adjoints K^* and L^* of K and L and then he showed that $LL^* + K^*K = M$ is a tame bijection with a tame inverse.

Then $LL^*M^{-1} + K^*KM^{-1} = Id$. Now approximate solution to our linear problem for $(\tilde{f}, \tilde{g})$ can be defined as $\tilde{h} = L^*M^{-1}(S_t\tilde{f}, S_t\tilde{g})$ and $K^*KM^{-1}(S_t\tilde{f}, S_t\tilde{g})$ is proved to have a C_0 norm bounded by $Ct^\sigma \max\{\|\tilde{f}\|_0, \|\tilde{g}\|_0\}, \max\{\|\tilde{f}\|_1, \|\tilde{g}\|_1\}$. After this, Lemma 3 applies, along with the rest of the proof of Theorem 2.

Before we outline this proof we give some definitions.

Definition 2. Map $F : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is tame if there exists fixed numbers $\sigma > 0$ and $C > 0$ such that for all $r \geq 0$:

$$\|F\|_r \leq C\|F\|_{r+\sigma}$$

Definition 3. An exact sequence of maps $L : A \to B$, $K : B \to C$ ($KL = 0$) splits if there exist maps $K^* : C \to B$ and $L^* : B \to C$ such that $LL^* + K^*K = Id_B$. Then L^* and K^* is called a splitting for this exact sequence.

Lemma 4. *There exist tame operators $L*$, K^*, M and M^{-1} such that $LL^*M^{-1} + K^*KM^{-1} = Id_{C_0^\infty(\mathbb{T})^2}$, where $C_0^\infty(\mathbb{T})$ denotes the space of C^∞ functions of average 0.*

Proof: During the course of the proof we swich to notation (α_1, α_2) instead of (α, β) for simultaneously Diophantine numbers on the statement of the Theorem 5.

Definitions of L^* and K^*

Let $L_i v = v(x + \alpha_i) - v(x)$, and let $L_i^* v = v(x - \alpha_i) - v(x)$

Let $L^*(w_1, w_2) = L_1^* w_1 + L_2^* w_2$.

Let $K(w_1, w_2) = \begin{pmatrix} 0 & L_1w_2 - L_2w_1 \\ L_2w_1 - L_1w_2 & 0 \end{pmatrix}$

Let $K^*(\begin{pmatrix} z_{11} & z_{21} \\ z_{21} & z_{22} \end{pmatrix}) = (L_1^*z_{11} + L_2^*z_{21}, L_1^*z_{12} + L_2^*z_{22})$

Finally, let $M(w_1, w_2) = (LL^* + K^*K)(w_1, w_2)$. Using the fact that L_i and L_j^* commute, M is reduced to: $M(w_1, w_2) = (\mathbb{M}w_1, \mathbb{M}w_2)$, where $\mathbb{M} = L_1L_1^* + L_2^*L_2$. It is very simple to show that all the operators defined above are tame. in fact, no regularity is lost after application of any one of those operators defined above.

$\mathbb{M}$ is a bijection and it has a tame inverse M^{-1} on $C^\infty(\mathbb{T})$. To show this, look at M, namely

$$w = \mathbb{M}v = 4v(x) - v(x - \alpha_1) - v(x + \alpha_1) - v(x - \alpha_2) - v(x + \alpha_2) = \phi$$

Now by using Fourier expansions for w and v, the Fourier coefficients of w are:

$$\hat{w}_n = \hat{v}_n(4 - 2\cos 2\pi n\alpha_1 - 2\cos 2\pi n\alpha_2)$$

Since we assume $(\alpha_1, \alpha_2) \in SDC(C, \tau)$, we have: $|4 - 2\cos 2\pi n\alpha_1 - 2\cos 2\pi n\alpha_2| \geq |4(sin\pi n\alpha_1)^2 + 4(sin\pi n\alpha_2)^2| \geq \max \|n\alpha_1\|^2, \|n\alpha_2\|^2 \geq C|n|^{-2\tau}$, it follows that

$$\|v\|_r \leq C_r \|w\|_{r+\sigma}$$

where $\sigma = 2\tau + 2$.

□

Lemma 5. *Let $(\alpha, \beta) \in SDC(\tau, C)$, let $\tilde{f}, \tilde{g} \in C^\infty(\mathbb{T}, \mathbb{R})$ such that $f \circ g = g \circ f$ where $f = R_\alpha + \tilde{f}$ and $\mathcal{G} = R_\beta + \tilde{g}$. Then there exists $\tilde{h}$ such that:*

$$\|(L_\alpha \tilde{h}, L_\beta \tilde{h}) - (S_t \tilde{f} - [\tilde{f}], S_t \tilde{g} - [\tilde{g}])\|_0 \leq C t^\sigma \|\tilde{f}, \tilde{g}\|_0 \|\tilde{f}, \tilde{g}\|_1 \tag{1.20}$$

and

$$\|\tilde{h}\|_r \leq C_r t^\sigma \|\tilde{f}\|_r \tag{1.21}$$

for all $r \geq 0$ and $\sigma > 2 + 2\tau$.

Proof: To shorten the notation, we denote $S_t \tilde{f} - [\tilde{f}]$ by $S_t \tilde{f}_0$ and $S_t \tilde{g} - [\tilde{g}]$ by $S_t \tilde{g}_0$ By applying Lemma 4 to $(S_t \tilde{f}_0, S_t \tilde{g}_0)$ we have:

$$LL^* M^{-1}(S_t \tilde{f}_0, S_t \tilde{g}_0) + K^* K M^{-1}(S_t \tilde{f}_0, S_t \tilde{g}_0) = (S_t \tilde{f}_0, S_t \tilde{g}_0)$$

So we can define

$$\tilde{h} := L^* M^{-1}(S_t \tilde{f}_0, S_t \tilde{g}_0)$$

Then

$$\|\tilde{h}\|_r \leq C_r \|S_t \tilde{f}, S_t \tilde{g}\|_{r+\sigma} \leq C_r t^\sigma \|\tilde{f}, \tilde{g}\|_r$$

Now to check the main inequality, since K^* is tame with no loss of regularity, L_i^* and L_k commute, and $S_t(f \circ R_\alpha) = (S_t f) \circ R_\alpha$ (smoothing operators can be chosen so that they are translation invariant [6], **[?]**):

(1.22)

$$\begin{aligned}
\|L\tilde{h} - (S_t \tilde{f}_0, S_t \tilde{g}_0)\|_0 &= \|K^* K M^{-1}(S_t \tilde{f}_0, S_t \tilde{g}_0)\|_0 \leq \|K(\mathbb{M}^{-1}(S_t \tilde{f}_0), \mathbb{M}^{-1}(S_t \tilde{g}_0))\|_0 \\
&\leq C\|\mathbb{M}^{-1} S_t K_{12}(\tilde{f}_0, \tilde{g}_0)\|_0 \leq C\|S_t K(\tilde{f}_0, \tilde{g}_0)\|_\sigma \leq C t^\sigma \|K(\tilde{f}_0, \tilde{g}_0)\|_0 \\
&\leq C t^\sigma \|\tilde{f}, \tilde{g}\|_0 \|\tilde{f}, \tilde{g}\|_1
\end{aligned}$$

□

1.4.3. *Global result for $\mathbb{Z}^k$ actions on $\mathbb{T}$.*

Theorem 6. *If $(\alpha_1, \alpha_2) \in SDC(\tau, C)$ then any commuting pair $(f, g) \in F^{\infty}_{(\alpha,\beta)}(\mathbb{T})$ is simultaneously and smoothly conjugated to (R_α, R_β).*

2. TORAL AUTOMORPHISMS

2.1. **Set-up, basic facts and some heuristics.** We consider a $\mathbb{Z}^2$ action by automorphisms of $\mathbb{T}^N$. This action is defined by a map $\alpha : \mathbb{Z}^2 \to SL(N, \mathbb{Z})$, and for $\alpha(n)(x)$ we will sometimes use a notation $\alpha(n, x)$. Let us call the generating matrices A and B and denote this action also by $\alpha :< F_A, F_B >$, or sometimes simply as $\alpha :< A, B >$ keepinf in mind that it takes place on the torus.

First, it is possible that such an action factors to an action which is essentially generated by a single diffeomorphism. Namely, an action $\alpha' : \mathbb{Z}^2 \times \mathbb{T}^{N'} \to \mathbb{T}^{N'}$ is an *algebraic factor* of α if there exists an epimorphism $h : \mathbb{T}^N \to \mathbb{T}^{N'}$ such that $h \circ \alpha = \alpha' \circ h$.

An action α' is a *rank-one factor* if it is an algebraic factor and if $\rho_{\alpha'}(\mathbb{Z}^k)$ contains a cyclic subgroup of finite index.

If it so happens that α has a rank-one factor than we cannot expect to have rigidity. Single toral automorphism, if hyperbolic, is structurally stable but its eigenvalues serve as moduli of smooth conjugacy. If a single toral automorphism is partially hyperbolic, then it is not even structurally stable.

However, if we assume that α is such that *every non-trivial element of α is ergodic* then α has no non-trivial rank-one factors. This fact was explicitely proved by Starkov [7] but was well known before his proof came about. The proof exploits the fact that ergodicity for toral automorphisms just means that the corresponding matrix has no eigenvalues which are roots of unity.

Proposition 3. *(exercise) Let A be a matrix in $SL(N, \mathbb{Z})$ and let F_A denote the corresponding map on $\mathbb{T}^N$. The following are equivalent:*
(1) F_A is ergodic
(2) A has no eigenvalues which are roots of unity
(3) There are no finite orbits for the dual action of A
(4) F_A is mixing

Thus from now on we assume that the action $\alpha :< A, B >$, $AB = BA$, is such that for any $(k, l) \in \mathbb{Z}^2 \setminus 0$ we have that $A^k B^l$ is ergodic.

Now assume that we have another $\mathbb{Z}^2$ action $\tilde{\alpha}$ which is C^∞ and close to α in some, say C^l toplogy. In other words (after passing to the lifts) we consider a $\mathbb{Z}^2$ action by diffeomorphisms of $\mathbb{T}^N$, generated by two commuting maps $A + R_A$ and $B + R_B$ such that $R_A, R_B \sim \epsilon << 1$ in some C^l norm. As for Anosov actions, the question is whether this action $\tilde{\alpha}$ is smoothly conjugated to α providing ϵ is sufficiently small i.e. whether there exists a C^∞ map $H = Id + \Omega$ of $\mathbb{T}^N$ so that

$$\begin{aligned} H \circ A &= (A + R_A) \circ H \\ H \circ B &= (B + R_A) \circ H \end{aligned} \tag{2.1}$$

This is our non-linear problem. As before we linearize it and ask two questions:

Q1: What are the obstructions to solving the linear problem?

Q2: Assuming that the obstructions vanish, how well can we solve the linear problem?

If answers to either one of these questions are not within reach than we have no chance of applying the iteration scheme similar to that applied for circular rotations. If we can answer these questions than we can hope to be able to get into the iteration process at least.

The linear problem is:

$$\begin{aligned} A\Omega - \Omega \circ A &= R_A \\ B\Omega - \Omega \circ B &= R_B \end{aligned} \tag{2.2}$$

After choosing appropriate coordinates (and assuming for the moment that A and B have no non-trivial Jordan blocks), the equations above simplify to pairs of equations of the kind:

$$\begin{aligned} L_{\lambda,A}(\omega) &:= \lambda\theta - \theta \circ A = \theta \\ L_{\mu,B}(\omega) &:= \mu\theta - \theta \circ B = \psi \end{aligned} \tag{2.3}$$

where λ and μ are corresponding eigenvalues of A and B, so they ar not roots of unity, and θ and ψ are C^∞ functions on $\mathbb{T}^N$.

Then one obstruction is easy to notice. If there is a solution ω, then

Obstructions-2: $L_{\lambda,A}(\psi) - L_{\mu,B}(\theta) = 0$

Since λ and μ are not 1, there are no other obvious obstructions (such as average).

Now we address the second question. Assume we do not have the obstruction above, i.e assume that indeed $L_{\lambda,A}(\psi) = L_{\mu,B}(\theta)$. Then the question is how to construct a solution.

For a single equation $\lambda\omega - \omega \circ A = \theta$ we can look at the dual problem and get $\lambda\hat{\omega}_n - \hat{\omega}_{A^*n} = \hat{\theta}_n$. By iterating forward (or backward) we get

$$\hat{\omega}_n^{\overset{+}{(-)}} = \overset{+}{(-)} \sum_{\substack{i \geq 0 \\ (i \leq -1)}} \lambda^{-(i+1)} \hat{\theta}_{A^i n}, \qquad n \neq 0.$$

Each sum converges absolutely since θ is C^∞ and all non-zero integer vectors have non-trivial projections to expanding and contracting subspaces for A due to the ergodicity assumption on A.

Clearly, if solution ω exists, the two sums, positive and negative, ought to be the same. In other words we should have that

$$\sum_{\lambda, A} \hat{\theta}_n := \sum_{i \in \mathbb{Z}} \lambda^{-(i+1)} \hat{\theta}_{A^i n} = 0, \; n \in \mathbb{Z}^N \setminus 0$$

So it seems that there are in fact infinitely many obstructions in the dual space to solving each one of the equations in (2.2).

However, if we assume that (Obstruction-3) vanishes, then all of these obstructions for single equations vanish as well (Lemma 6). Once the obstructions for single equations vanish, the equations can be solved with good estimates. This is the content of the subsequent section.

After that we will describe how to get rid of small obstructions and at the end we sketch a proof of local rigidity for higher rank ergodic actions by toral automorphisms.

2.2. **Cocycle rigidity.**

Lemma 6. *If $K(\psi, \theta) = L_{\lambda,A}(\psi) - L_{\mu,B}(\theta) = 0$, for C^∞ functions ψ and θ, then the obstructions to solving equations:*

$$\begin{aligned} L_{\lambda,A}(\omega) &= \theta \\ L_{\mu,B}(\omega) &= \psi \end{aligned} \tag{2.4}$$

vanish.

Proof:

$$K(\theta, \psi) = L_{\mu,B}\theta - L_{\lambda,A}\psi = 0 \tag{2.5}$$

After passing to to the dual action this implies:

$$\begin{aligned} \sum_{\mu,B} (L_{\mu,B}\hat{\theta}_n) &= \sum_{\mu,B} (L_{\lambda,A}\hat{\psi}_n) \\ \sum_{\lambda,A} (L_{\lambda,A}\hat{\psi}_n) &= \sum_{\lambda,A} (L_{\mu,B}\hat{\theta}_n) \end{aligned} \tag{2.6}$$

Consider now the first equation above. Since all the sums involved converge absolutely we have

$$L_{\mu,B}\sum_{\mu,B}\hat{\theta}_n = 0$$

which implies

$$\sum_{\mu,B}(L_{\lambda,A}\hat{\psi}_n) = 0$$

This implies that the obstruction for ψ is not only multiplied by μ under the action of B, but is also multiplied by λ under the action of A, i.e. $\lambda\sum_{\mu,B}\hat{\psi}_n = \sum_{\mu,B}\hat{\psi}_{An}$. By iterating this equation we obtain:

$$\lambda^k\sum_{\mu,B}\hat{\psi}_n = \sum_{\mu B}\hat{\psi}_{A^k n},$$

for every $k \in \mathbb{Z}$. Therefore

$$\sum_{k\in\mathbb{Z}}\lambda^k\sum_{\mu,B}\hat{\psi}_n = \sum_{k\in\mathbb{Z}}\sum_{\mu B}\hat{\psi}_{A^k n} \tag{2.7}$$

The series in the left hand side of (2.7) does not converge unless $\sum_{\mu B}\hat{\psi}_n = 0$ while the right hand side of (2.7) converges absolutely by Lemma 9 b). Therefore $\sum_{\mu B}\hat{\psi}_n = 0, \forall n \neq 0$. Similarly $\sum_{\mu A}\hat{\theta}_n = 0, \forall n \neq 0$. $\square$

Now we consider a single equation: $\lambda\omega - \omega \circ A = \phi$, we assume that the obstructions for ϕ vanish and ask how we can solve for ω. This is done in Lemma 8. However, before we state this and prove it, we state the following fact which is very useful:

Lemma 7. *(Katznelson,* [5]*) Let A be an $N \times N$ matrix with integer coefficients. Assume that $\mathbb{R}^N$ splits as $\mathbb{R}^N = V \oplus V'$ with V, V' invariant under A and such that $A|_V$ and $A|_{V'}$ have no common eigenvalues. If $V \cap \mathbb{Z}^N = \{0\}$ then there exists a constant γ such that $d(n, V) \geq \gamma\|n\|^{-N}$ for all $n \in \mathbb{Z}^N$ where $\|\cdot\|$ is Euclidean norm and d is Euclidean distance.*

Remark 9. This can be viewed as a version of the Liouville's theorem about rational approximation of algebraic irrationals, i.e.$|\alpha - \frac{m}{n}| \geq Cn^{-N}$ for any non-zero integers m and n, where α is an irrational first order root of an integer polynomial of degree N. The proof of this classical result inspires the proof of Lemma 7 in [5] which we repeat here since it gives some insight on arithmetic vs. dynamical properties of toral automorphisms.

Proof: Any polynomial p sufficiently close to the minimal polynomial f of A on V satisfies the condition $p(A)n \neq 0$ for all $n \in \mathbb{Z}^N, n \neq 0$ because its null space is contained in V and $V \cap \mathbb{Z}^N = \{0\}$ by assumption. Then one can construct a polynomial f_Q with rational coefficients of that kind, and since A is an integer matrix we have $\|f_Q(A)n\| > \frac{1}{q}$ (the choice is made as $|a_j - r_j/q| \leq \frac{1}{qQ}$, where a_j are coefficients of f, r_j/q coefficients of f_Q and $q \leq Q^k$) for any non-zero n. Now if n_V is the projection of n to V then:

$$f_Q(A)n = f_Q(A)(n - n_V) + (f_Q(A) - f(A))n_V$$

This implies $\frac{1}{q} \leq C(d(n,V) + \frac{\|n\|}{qQ})$. Then by choosing $Q = C\|n\|$ where C is a constant depending on A, the estimate follows:

$$d(n,V) > \frac{1}{Cq} \geq \frac{1}{CQ^k} > C_1\|n\|^{-k} > C_1\|n\|^{-N}$$

with C_1 being a positive constant depending only on A. $\square$

Remark 10. In particular, if A is ergodic the dual map A^* on $\mathbb{Z}^N$ induces a decomposition of $\mathbb{R}^N$ into expanding, neutral and contracting subspaces. We will denote the expanding subspace by $V_1(A)$, the contracting subspace by $V_3(A)$ and the neutral subspace by $V_2(A)$.

$$\mathbb{R}^N = V_1(A) \oplus V_2(A) \oplus V_3(A) \tag{2.8}$$

All three subspaces $V_i(A)$, $i = 1,2,3$ are A-invariant and

$$\begin{aligned} &\|A^i v\| \geq C\rho^i \|v\|, \quad \rho > 1, \quad i \geq 0, \quad v \in V_1(A) \\ &\|A^i v\| \geq C\rho^{-i} \|v\|, \quad \rho > 1, \quad i \leq 0, \quad v \in V_3(A) \\ &\|A^i v\| \geq C(|i|+1)^{-N} \|v\|, \quad i \in \mathbb{Z}, \quad v \in V_2(A) \end{aligned} \tag{2.9}$$

In Lemma 7 let $V = V_3 \oplus V_2$ from then $V \cap \mathbb{Z}^N = \{0\}$. So Lemma 7 implies for $n \in \mathbb{Z}^N$:

$$\|\pi_1(n)\| \geq \gamma \|n\|^{-N}$$

where $\pi_1(n)$ is the projection of n to V_1, the expanding subspace for A.

Lemma 8. *Let θ be a C^∞ function on the torus and $\lambda \in \mathbb{C}$, $\lambda \neq 1$. Let A be an integer matrix in $GL(N, \mathbb{Z})$ defining an ergodic automorphism of $\mathbb{T}^N$ such that for all non-zero $n \in \mathbb{Z}^N$ the following sums along the dual orbits are zero i.e.*

$$\sum_{i=-\infty}^{+\infty} \lambda^{-(i+1)} \hat{\theta}_{A^i n} = 0 \tag{2.10}$$

then the equation:

$$\lambda\omega - \omega \circ A = \theta \tag{2.11}$$

has a C^∞ *solution* ω *and the following estimate:*

$$\|\omega\|_{a-\delta} \le \frac{C_a}{\delta^\nu}\|\theta\|_a \tag{2.12}$$

holds for $\delta > 0, \nu = aN + 1$ *and* $a > \frac{|\log|\lambda||}{\log\rho}$. *Here* $\rho > 1$ *is the expansion rate for* A *from* (2.9). *Thus for* $r \ge 0$*:*

$$\|\omega\|_{C^r} \le C_r\|\theta\|_{C^{r+\sigma}} \tag{2.13}$$

where σ *is an integer greater than* $\max\{N+1, \frac{|\lg|\lambda||}{\lg\rho}\}$.

Proof: Suppose ω is a C^∞ solution to the and let $\hat{\omega}_n$ and $\hat{\theta}_n$ denote Fourier coefficients of ω and θ. Then the equation $\lambda\omega - \omega \circ A = \theta$ in the dual space has the form:

$$\lambda\hat{\omega}_n - \hat{\omega}_{An} = \hat{\theta}_n, \quad \forall n \in \mathbb{Z}^N$$

For $n = 0$, since $\lambda \neq 1$, we can immediately calculate $\hat{\omega}_0 = \frac{\hat{\theta}_0}{\lambda - 1}$. For $n \neq 0$ the dual equation has two solutions

$$\hat{\omega}_n^{\overset{+}{(-)}} = \overset{+}{(-)} \sum_{\substack{i \ge 0 \\ (i \le -1)}} \lambda^{-(i+1)}\hat{\theta}_{A^i n}, \qquad n \neq 0.$$

Each sum converges absolutely since θ is C^∞ and all non-zero integer vectors have non-trivial projections to expanding and contracting subspaces for A due to the ergodicity assumption on A. By assumption $\sum^{\lambda,A}\hat{\theta}_n = 0$, $\forall n \neq 0$ i.e. $\hat{\omega}_n^+ = \hat{\omega}_n^- \overset{\text{def}}{=} \hat{\omega}_n$. This gives a formal solution $\omega = \sum \hat{\omega}_n^+ e_n = \sum \hat{\omega}_n^- e_n$. We estimate each $\hat{\omega}_n$ using both of its forms in order to show that ω is C^∞.

If n is mostly contracting, i.e. if $n \hookrightarrow 3(A)$ use the $\hat{\omega}_n^-$ form for the solution to obtain the following bound on n-th Fourier coefficient:

$$\begin{aligned}
|\hat{\omega}_n| = \left|\sum_{k \le 0} \lambda^{-(k+1)}\hat{\theta}_{A^k n}\right| &\le \sum_{k \le 0} |\lambda|^{-(k+1)}|\hat{\theta}_{A^k n}| \le \|\theta\|_a \sum_{k \le 0} |\lambda|^{-(k+1)}|A^k n|^{-a} \\
&\le \|\theta\|_a \sum_{k \le 0} |\lambda|^{-(k+1)}\|A^k \pi_3(n)\|^{-a} \le \|\theta\|_a C^{-a} \sum_{k \le 0} |\lambda|^{-(k+1)}\rho^{ak}\|\pi_3(n)\|^{-a} \\
&\le C_a\|\theta\|_a|n|^{-a}
\end{aligned}$$

where $a > \frac{\log|\lambda|}{\log\rho}$.

Similarly, if $n \hookrightarrow 1(A)$, using the form $\hat{\omega}_n = \hat{\omega}_n^+$, the estimate $|\hat{\omega}_n| \le C_a \|\theta\|_a |n|^{-a}$ holds, if $a > \frac{\log|\lambda|^{-1}}{\log\rho}$.

If $n \hookrightarrow 2(A)$ and $|\lambda| \ge 1$ using the form $\hat{\omega}_n^+$ of the solution it follows that:

$$
\begin{aligned}
|\hat{\omega}_n| &\le \|\theta\|_a \sum_{k\ge 0} |\lambda|^{-(k+1)} |A^k n|^{-a} \\
&\le \|\theta\|_a C^{-a} \sum_{k\ge 0} |\lambda|^{-(k+1)} (1+k)^{N\alpha} \|\pi_2(n)\|^{-a}
\end{aligned} \tag{2.14}
$$

However, the above sum need not converge. This is where we use again the fact that A is ergodic. Namely, no integer vector can stay mostly in the neutral direction for too long. After the time which is approximately $\lg|n|$ the expanding direction takes over. The precise statement of this fact is Lemma 7. Namely, from Lemma 7 it follows that $\|\pi_1(n)\| \ge \gamma |n|^{-N}$ for some γ and all n. Therefore

$$|A^k n| \ge \|A^k \pi_1(n)\| \ge C\rho^k \|\pi_1(n)\| \ge \gamma C \rho^k |n|^{-N} \ge \gamma C \rho^{k-k_0} |n|$$

for $k \ge k_0$ and $k_o = [\frac{(1+N)\log|n|}{\log\rho}] + 1$. This fact can be used to estimate most terms of the series in (2.14), that is, all but finitely many terms. For the rest the polynomial estimate in (2.9) for vectors in V_2 holds. Hence:

$$|\hat{\omega}_n| \le \|\theta\|_a \sum_{k=0}^{k_0-1} |\lambda|^{-(k+1)} |k|^{Na} \|\pi_2(n)\|^{-a} + C\|\theta\|_a \sum_{k=k_0}^{\infty} |\lambda|^{-(k+1)} \rho^{-a(k-k_0)} |n|^{-a}$$

Thus using that $n \hookrightarrow 2(A)$ and $|\lambda| > 1$ we have:

$$|\hat{\omega}_n| \le C\|\theta\|_a |k_0|^{Na+1} |n|^{-a} + C\|\theta\|_a |n|^{-a}$$

Now by choice of k_0, $k_0 \sim \log|n|$. Thus the following estimate holds:

$$|\hat{\omega}_n| \le C_a (\log|n|)^{Na+1} |n|^{-a} \|\theta\|_a$$

For $|\lambda| < 1$ the same estimate follows using the second form for $\hat{\omega}_n$ i.e. the negative sum and the fact that A^{-1} is also an ergodic toral automorphism thus going backwards in time the contracting direction takes over, that is, we use the Lemma 7 for A^{-1}. Therefore for all $n \in \mathbb{Z}^N$ we have:

$$|\hat{\omega}_n|\, |n|^{a-\delta} \le \frac{C_a}{\delta^\nu} \|\theta\|_a$$

for $a > \frac{|\log|\lambda||}{\log \rho}$ and $\delta > 0$. This implies the estimate (2.12) for $\|\omega\|_{a-\delta}$. The estimate for C^r norms follows immediately using the norm comparison from Section **??**. In particular if θ is C^∞, ω is also C^∞. $\square$

It is clear that from the above two statements we get a corollary:

Corollary 2. *If $K(\psi, \theta) = L_{\lambda,A}(\psi) - L_{\mu,B}(\theta) = 0$, for C^∞ functions ψ and θ, then equations:*

$$L_{\lambda,A}(\omega) = \theta L_{\mu,B}(\omega) = \psi \tag{2.15}$$

have a common C^∞ solution which also satisfies the estimate:

$$\|\omega\|_{C^r} \leq C_r \|\theta, \psi\|_{C^{r+\sigma}} \tag{2.16}$$

(2.17)

2.3. **Orbit growth for the dual action.** In this Section the crucial estimates for the exponential growth along individual orbits of the dual action are obtained. They may be viewed as a generalization of Lemma 7 to higher rank actions by toral automorphisms.

The existence of such estimates in case of $\mathbb{Z}^d$ actions with $d \geq 2$ relies fundamentally on the higher rank assumption i.e. on the ergodicity of all non-trivial elements of the action.

Lemma 9. *Let α be a $\mathbb{Z}^2$ action by automorphisms of $\mathbb{T}^N$ such that all non-trivial elements of the action are ergodic. Then there exist constants $\tau > 0$ and $C > 0$ depending on the action only, such that:*

a) For every integer vector $n \in \mathbb{Z}^N$ and for all $k \in \mathbb{Z}^2$:

$$|\alpha^k n| \geq C \exp\{\tau \|k\|\} |n|^{-N}$$

b) For any C^∞ function φ on the torus, any non-zero $n \in \mathbb{Z}^N$ and any vector $y \in \mathbb{R}^2$ the following sums

$$S_K(\varphi, n) = \sum_{k \in K} y^k \hat{\varphi}_{\alpha^k n},$$

where $y^k \stackrel{\text{def}}{=} \prod_{i=1}^2 y_i^{k_i}$, converge absolutely for any $K \subset \mathbb{Z}^2$.

c) Assume in addition to the above that for an $n \in \mathbb{Z}^N$ and for every $k \in K = K(n) \subset \mathbb{Z}^2$ we have $P(\|k\|)|\alpha^k n| \geq |n|$ where P is a polynomial of degree N. Then:

$$|S_K(\varphi, n)| \leq C_{a,y} \|\varphi\|_a |n|^{-a+\kappa_{y,\alpha}} \tag{2.18}$$

for any $a > \kappa_{y,\alpha} \stackrel{\text{def}}{=} \frac{N+1}{\tau} \sum_{i=1}^2 |\log|y_i||$.

d) If the assumptions of c) are satisfied for every $n \in \mathbb{Z}^N$ then the function

$$S(\varphi) \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z}^N} S_{K(n)}(\varphi, n) e_n$$

is a C^∞ function if φ is. Moreover, the following norm comparison holds:

$$\|S(\varphi)\|_{C^r} \leq C_{r,y} \|\varphi\|_{C^{r+\sigma}} \tag{2.19}$$

for $r > 0$ and any $\sigma > N + 2 + [\kappa_{y,\alpha}]$.

Proof: *Proof of a).*

we first notice that it is enough to obtain the constant τ and to show the exponential estimate in a) in the semisimple case i.e. when the action is generated by matrices A_1, A_2 which are simultaneously diagonalizable over $\mathbb{C}$. If the action is not semisimple, only polynomial growth may occur in addition, thus the same estimate holds with slightly smaller τ and with possibly larger C (for more deatails see [**?**])

Here we give the proof in the case when the action is irreducible which shows the main idea, but is technically simpler and we refer to [3] for the proof in the general case.

In the case when the action is irreducible we may project a non-trivial $n \in \mathbb{Z}^N$ to the Lyapunov directions corresponding to non-zero Lyapunov exponents of the action. Lyapunov exponents are defined as:

$$\chi_i(k) = \sum_{j=1}^{2} k_j \ln |\lambda_{ij}|$$

where $k = (k_1, k_2) \in \mathbb{Z}^2$, $i = 1, .., r$, and $\lambda_{1j}, ..\lambda_{rj}$ are the eigenvalues of A_j for $j = 1, 2$. Individual Lyapunov directions are irrational and due to the irreducibility assumption each of the projections of the vector n to the Lyapunov directions is non-trivial. Thus one may apply the Katznelson lemma to each of these projections and choose τ as the minimum of the function $\max \chi_i(t)$ for t on the unit sphere in $\mathbb{R}^2$. This minimum has to be positive.

Indeed, if the minimum was non-positive, it would have to be zero since $\sum_i \chi_i(t) = 0$ for all t. This implies that, since t is not zero, all Lyapunov lines coincide. In other words, all non-zero Lyapunov exponents are proportional. Thus, if the action has non-trivial neutral direction, it would have to contain elements which have all Lyapunov exponents zero, and this is only possible for integer matrices if all the eigenvalues are roots of unity which contradicts the ergodicity assumption. Therefore the neutral direction is trivial, and in all the other directions Lyapunov exponents are proportional, so this is in fact a rankone action, generated by a single toral automorphism. Therefore, τ is strictly positive.

Thus using the following norm:

$$\|\alpha^k n\|_\chi = \sum_{i=1}^{r} \|n_i\| \exp \chi_i(k)$$

where n_i are projections of n to the corresponding Lyapunov directions, and the Katznelson's Lemma, we obtain the needed estimate

$$|\alpha^k n| \geq C\|\alpha^k n\|_\chi \geq C \exp\{\tau\|k\|\} \min_i \|n_i\| \geq C \exp\{\tau\|k\|\}\|n\|^{-N}.$$

The statements b)-d) are left as an exercise.

□

2.4. **Approximating almost a cocycle by a cocycle.** Notation: Given a complex number λ and a function φ on the torus, define the twisted coboundary operators:

$$\Delta_A^\lambda \varphi \stackrel{\text{def}}{=} \lambda\varphi - \varphi \circ A \tag{2.20}$$

$$\Delta_A^\lambda \hat{\varphi}_n \stackrel{\text{def}}{=} \lambda\hat{\varphi}_n - \hat{\varphi}_{An}, \tag{2.21}$$

In what follows λ will usually be an eigenvalue of A, and μ will usually denote an eigenvalue of B, so we will often use the following simpler notation:

$$\Delta^\lambda \stackrel{\text{def}}{=} \Delta_A^\lambda, \quad \Delta^\mu \stackrel{\text{def}}{=} \Delta_B^\mu$$

$$\Delta^\lambda \hat{\varphi}_n \stackrel{\text{def}}{=} \Delta_A^\lambda \hat{\varphi}_n, \quad \Delta^\mu \hat{\varphi}_n \stackrel{\text{def}}{=} \Delta_B^\mu \hat{\varphi}_n$$

Lemma 10. *Let θ, ψ, φ be C^∞ functions such that $L(\theta,\psi) = \Delta^\mu\theta - \Delta^\lambda\psi = \varphi$, then it is possible to split θ and ψ as*

$$\theta = \mathcal{P}\theta - \mathcal{E}\theta$$
$$\psi = \mathcal{P}\psi + \mathcal{E}\psi$$

so that $L(\mathcal{P}\theta, \mathcal{P}\psi) = 0$, $L(\mathcal{E}\theta, \mathcal{E}\psi) = \varphi$ and the following estimates hold:

$$\|\mathcal{E}\theta, \mathcal{E}\psi\|_{C^r} \leq C\|\varphi\|_{C^{r+\sigma}} \tag{2.22}$$

for any $r > 0$ and any $\sigma > \tilde{M}_{\lambda,\mu}$ and

$$\|\mathcal{P}\theta, \mathcal{P}\psi\|_{C^r} \leq C\|\theta,\psi\|_{C^{r+\sigma}} \tag{2.23}$$

for any $r > 0$ and any $\sigma > \dot{M}_{\lambda,\mu}$. As λ and μ are eigenvalues of A and B, constants $\tilde{M}_{\lambda,\mu}$ and $\dot{M}_{\lambda,\mu}$ depend only on A, B and the dimension of the torus and are precisely defined below (see (**??**) *and* (2.35)*).*

Proof: *(i) Construction of $\mathcal{P}\theta, \mathcal{P}\psi, \mathcal{E}\theta$ and $\mathcal{E}\psi$.*

Let $\omega \stackrel{\text{def}}{=} \sum \hat{\omega}_n e_n$ where

$$\hat{\omega}_n \stackrel{\text{def}}{=} \begin{cases} \sum_{+}^{B} \hat{\psi}_n, & n \hookrightarrow 1,2(B) \\ -\sum_{-}^{B} \hat{\psi}_n, & n \hookrightarrow 3(B) \end{cases}$$

for $n \in \mathbb{Z}^N \setminus \{0\}$ and $\hat{\omega}_0 = (\mu - 1)^{-1} \hat{\psi}_0$.

Let

$$\mathcal{P}\psi \stackrel{\text{def}}{=} \Delta^{\mu}\omega = \mu\omega - \omega \circ B \tag{2.24}$$

Call n minimal and denote it by n_{min} if n is the lowest point on its B-orbit in the sense that $n \hookrightarrow 3(B)$ and $Bn \hookrightarrow 1,2(B)$ (for the definition of "$\hookrightarrow$" see Section **??**). There is one such minimal point on each non-trivial dual B-orbit. Now let $\mathcal{E}\psi \stackrel{\text{def}}{=} \sum \widehat{\mathcal{E}\psi_n} e_n$ where

$$\widehat{\mathcal{E}\psi_n} \stackrel{\text{def}}{=} \begin{cases} \mu \sum^{B} \hat{\psi}_n, & n = n_{min} \\ 0, & \text{otherwise} \end{cases} \tag{2.25}$$

for $n \neq 0$ and $\widehat{\mathcal{E}\psi_0} \stackrel{\text{def}}{=} 0$. Then it is easy to check that

$$\psi = \mathcal{P}\psi + \mathcal{E}\psi$$

In part (ii) and (iii) bellow we will show that both $\mathcal{P}\psi$ and $\mathcal{E}\psi$ are smooth functions such that $\mathcal{P}\psi$ is of the order of ψ and $\mathcal{E}\psi$ is the order of φ.

Let us define $\mathcal{P}\theta$ as:

$$\mathcal{P}\theta \stackrel{\text{def}}{=} \Delta^{\lambda}\omega \tag{2.26}$$

Then it is easy to see that $L(\mathcal{P}\theta, \mathcal{P}\psi) = 0$ since operators Δ^{λ} and Δ^{μ} commute due to the commutativity of the generators A and B. Therefore by defining $\mathcal{E}\theta$ as:

$$\mathcal{E}\theta \stackrel{\text{def}}{=} \theta - \mathcal{P}\theta \tag{2.27}$$

we obtain $L(\mathcal{E}\theta, \mathcal{E}\psi) = \varphi$ i.e.

$$\Delta^{\mu}\mathcal{E}\theta = \Delta^{\lambda}\mathcal{E}\psi + \varphi \tag{2.28}$$

Since operators Δ^{μ} and Δ^{λ} are bounded, if $\mathcal{E}\psi$ is proved to be smooth with norm comparable to some norm of φ, then by Lemma 8 the same holds true for $\mathcal{E}\theta$ as a solution of the equation (2.28) (The operator Δ^μ is injective on C^∞ whenever $\mu \neq 1$. This fact is contained in the proof of the Lemma **??** and is a consequence of the ergodicity of B).

(ii) Estimates for $\mathcal{E}\psi$ and $\mathcal{E}\theta$.

To estimate $\mathcal{E}\psi$ we need to bound $\sum^B \hat{\psi}_n$ in case $n \hookrightarrow 3(B)$ and $Bn \hookrightarrow 1, 2(B)$ with respect to φ. Since $\Delta^\mu \theta = \Delta^\lambda \psi + \varphi$, the obstructions for $\Delta^\lambda \psi + \varphi$ with respect to B vanish, therefore:

$$\Delta^\lambda \sum^B \hat{\psi}_n = -\sum^B \hat{\varphi}_n$$

Iterating this equation with respect to A we obtain:

$$\sum^B \hat{\psi}_n + \lambda^{-l} \lim_{l\to\infty} \sum^B \lambda^{-l} \hat{\psi}_{A^l n} = -\sum_{i=0}^{l} \sum^B \hat{\varphi}_{A^i n}$$

From Lemma 9 b) the limit above is 0. By iterating backwards and applying the same reasoning, we obtain:

$$\sum^B \hat{\psi}_n = \sum_-^A \sum^B \hat{\varphi}_n = -\sum_+^A \sum^B \hat{\varphi}_n \tag{2.29}$$

In the notation of Lemma 9 c), (2.29) implies that for $n \in \mathbb{Z}^N$ which is minimal on its B orbit, we have:

$$\widehat{\mathcal{E}\psi_n} = S_{H^+}(\varphi, n) = -S_{H^-}(\varphi, n)$$

where H^+ is the set of lattice points (l, k) in $\mathbb{Z}^2$ with positive l, and H^- the set of points with negative l. Then according to Lemma 9 d), the needed estimate for $\mathcal{E}\psi$ with respect to φ follows if in at least one of the half-spaces H^- and H^+ the dual action satisfies some polynomial lower bound for every $n = n_{min}$.

In case $Bn \hookrightarrow 2(B)$ for all l and all k we obviously have:

$$|A^l B^k n| \geq C|l|^{-N}|k|^{-N}|n| \tag{2.30}$$

thus the polynomial estimate needed for the application of part c) of Lemma 9 is satisfied both in H^+ and H^- for such n.

However in the other case i.e. when $Bn \hookrightarrow 1(B)$ the same estimate holds either in H^+ or in H^-. This follows from the fact that in this case ($n \hookrightarrow 3(B)$ and $Bn \hookrightarrow 1(B)$), n is substantially large both in the expanding and in the contracting direction for B.

To see this we let n_{i_1} and n_{i_3} be (large) projections of n to some expanding and contracting Lyapunov subspaces V_{i_1} and V_{i_2} for B with Lyapunov exponents χ_{i_1} and χ_{i_2}, respectively, i.e. let

$$\|n_{i_1}\| \geq C|n| \qquad \text{and} \qquad \|n_{i_3}\| \geq C|n|$$

where C is some fixed positive number. Then (assuming for the moment that α is semisimple) this implies:

$$|A^l B^k n| \geq C \sum_{i=1}^{r} \exp \chi_i(l,k) \|n_i\| \tag{2.31}$$
$$\geq C(\exp \chi_{i_1}(l,k) + \exp \chi_{i_3}(l,k))|n|$$

Now we notice that the union H of half-spaces $\{(l,k) : \chi_{i_1}(l,k) \geq 0\}$ and $\{(l,k) : \chi_{i_3}(l,k) \geq 0\}$ covers either H^+ or H^-. Namely, for any $k \in \mathbb{Z}$, (l,k) is in H if $l(\frac{\log|\lambda_{i_1}|}{\log|\mu_{i_1}|} - \frac{\log|\lambda_{i_3}|}{\log|\mu_{i_3}|}) \geq 0$ and this is true for $l \geq 0$ or for $l \leq 0$ depending on the $\mathrm{sgn}(\frac{\log|\lambda_{i_1}|}{\log|\mu_{i_1}|} - \frac{\log|\lambda_{i_3}|}{\log|\mu_{i_3}|})$. Here $\lambda_{i_3}, \lambda_{i_1}$ and μ_{i_3}, μ_{i_1} are corresponding eigenvalues of A and B, respectively. Therefore, from (2.31) we obtain

$$|A^l B^k n| \geq C|n|$$

in H^+ or in H^- if α is semisimple. If α is not semisimple then it decomposes a product of its semisimple and its unipotent part. For the semisimple part we use the estimate above and in the unipotent part only a polynomial growth may occur. This implies that (2.30) holds in H^+ or in H^- for a general (non-semisimple) α.

Now choose the half-space in which the estimate (2.30) holds, that is choose one of the sums $S_{H^+}(\varphi, n)$ or $S_{H^-}(\varphi, n)$. then the assumptions of d) in Lemma 9 are satisfied for one of the sums above S_{H^+} or S_{H^-} and therefore the estimate for $\mathcal{E}\psi$ follows:

$$\|\mathcal{E}\psi\|_{a-\delta-\kappa_{(\lambda,\mu),\alpha}} \leq \frac{C_a}{\delta^\nu}\|\varphi\|_a \tag{2.32}$$

for any $a > \kappa_{(\lambda,\mu),\alpha}$ and any $\delta > 0$, where

$$\kappa_{(\lambda,\mu),\alpha} \stackrel{\text{def}}{=} \frac{N+1}{\tau}(|\log|\mu|| + |\log|\lambda||) \tag{2.33}$$

Here, $\tau = \tau(A, B) > 0$ is the constant chosen as in the Lemma 9 $a)$

As we mentioned in part (i), by construction we have $\Delta^\mu \mathcal{E}\theta = \Delta^\lambda \mathcal{E}\psi + \varphi$. This by using Lemma 8 implies the following estimate for $\mathcal{E}\theta$ with respect to φ:

$$\|\mathcal{E}\theta\|_{a-\delta-\kappa_{(\lambda,\mu),\alpha}} \leq \frac{C_a}{\delta^\nu}\|\varphi\|_a \tag{2.34}$$

for any $a > \kappa_{(\lambda,\mu),\alpha}$ and any $\delta > 0$. This implies the C^r estimate (2.22) for $\mathcal{E}\psi$ and $\mathcal{E}\theta$ with the loss $\sigma > \tilde{M}_{\lambda,\mu}$, where

$$\tilde{M}_{\lambda,\mu} \stackrel{\text{def}}{=} N + 2 + [\kappa_{(\lambda,\mu),\alpha}] \tag{2.35}$$

where $\kappa_{(\lambda,\mu),\alpha}$ is defined in (2.33).
□

2.5. **Linearization.** Let α be a linear action as described in Theorem **??**. Let $\tilde{\alpha}$ be its small perturbation (the topology in which the perturbation is made will become apparent from the proof). The goal is to prove the existence of a C^∞ map $\mathcal{H} : \mathbb{T}^N \to \mathbb{T}^N$ such that $\tilde{\alpha} \circ \mathcal{H} = \mathcal{H} \circ \alpha$.

One can consider the problem of finding a conjugacy as a problem of solving the following non-linear functional equation:

$$\mathcal{N}(\tilde{\alpha}, \mathcal{H}) \stackrel{\text{def}}{=} \tilde{\alpha} \circ \mathcal{H} - \mathcal{H} \circ \alpha = 0$$

Following the ideas of the elementary Newton method and assuming the existence of a linear structure in the neighborhood of the *id*, the *id* may be viewed as an approximate solution of the non-linear problem. The linearization of the operator $\mathcal{N}$ at (α, id) is:

$$\begin{aligned}\mathcal{N}(\tilde{\alpha}, H) &= \mathcal{N}(\alpha, id) + D_1\mathcal{N}(\alpha, id)(\tilde{\alpha} - \alpha) \\ &\quad + D_2\mathcal{N}(\alpha, id)(\Omega) + \operatorname{Res}(\tilde{\alpha} - \alpha, \Omega) \\ &= \tilde{\alpha} - \alpha + \alpha(\Omega) - \Omega \circ \alpha + \operatorname{Res}(\tilde{\alpha} - \alpha, \Omega)\end{aligned}$$

where $\Omega = H - id$, and $\operatorname{Res}(\tilde{\alpha} - \alpha, \Omega)$ is quadratically small with respect to $\tilde{\alpha} - \alpha$ and Ω. If one finds H so that the linear part of the equation above is zero, i.e.:

$$\alpha(H - id) - (H - id) \circ \alpha = -(\tilde{\alpha} - \alpha)$$

then such H is a better approximate solution of the equation $\mathcal{N}(\tilde{\alpha}, \mathcal{H}) = 0$ than the *id* is. After obtaining a better solution, the linearization procedure and solving the linearized equation may be repeated for the new perturbation leading to an even better approximation. The difficulties which arise in particular applications of this iterative scheme are of two kinds: one is to solve, or solve approximately the linearized equation, and the other has to do with obtaining good estimates for the solution so that the sequence of approximate solutions produced by this scheeme converges in some reasonable function space.

We now adapt this general scheme to our specific problem concerning toral automorphisms. Any map of the torus $\mathbb{T}^N$ into itself can be lifted to the universal cover $\mathbb{R}^N$. For every $g \in \mathbb{Z}^k$, the lift of $\alpha(g)$ is a linear map of $\mathbb{R}^N$ i.e. a matrix with integer entries and with determinant ± 1, which is also denoted by $\alpha(g)$. The lift of $\tilde{\alpha}(g)$

is $\alpha(g) + R(g)$ where $R(g)$ is an $N-$periodic map for every g, i.e. $R(g)(x+m) = R(g)(x)$ for $m \in \mathbb{Z}^N$. The lift of H is id $+ \blacksquare$ with an $N-$periodic Ω.

In terms of Ω the conjugacy equation is:

$$\alpha\Omega - \Omega \circ \alpha = -R \circ (\mathrm{id} + \blacksquare) \tag{2.36}$$

If Ω is a solution for the corresponding linearized equation

$$\alpha\Omega - \Omega \circ \alpha = -R \tag{2.37}$$

or at least its approximate solution i.e. if it solves (2.37) with an error which is small with respect to R, then one may expect that the new perturbation defined by:

$$\tilde{\alpha}^{(1)} \stackrel{\text{def}}{=} H^{-1} \circ \tilde{\alpha} \circ H$$

is much closer to α than $\tilde{\alpha}$ i.e. the new error

$$R^{(1)} \stackrel{\text{def}}{=} \tilde{\alpha}^{(1)} - \alpha$$

is expected to be small with respect to the old error R.

Also, if common solution to 2.37 exists, then :

$$L(R_A, R_B) \stackrel{\text{def}}{=} (R_A \circ B - BR_B) - (R_B \circ A - AR_B) = 0, \tag{2.38}$$

In terms of R this condition is saying that R is a *twisted cocycle* over the $\mathbb{Z}^2$ action on $\mathbb{T}^N$ generated by A and B.

The key step is to show that if R is *almost* a twisted cocycle over α then R is *close* to an actual twisted cocycle over α. In other words, we construct a projection $\mathcal{P}R$ of R to the space of twisted cocycles over α so that the difference $\mathcal{E}R = \mathcal{P}R - R$ is small with respect to R.

After this the rest of the proof goes along the same lines as the proof of Moser's theorem, including the smoothing.

3. General outline for applying the iteration scheme to group actions

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