Super-potentials for currents on compact Kähler manifolds and dynamics of automorphisms

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Abstract

We introduce a notion of super-potential (canonical function) associated to positive closed (p, p)-currents on compact Kähler manifolds and we develop a calculus on such currents. One of the key points in our study is the use of deformations in the space of currents. As an application, we obtain several results on the dynamics of holomorphic automorphisms: regularity and uniqueness of the Green currents. We also get the regularity, the entropy, the ergodicity and the hyperbolicity of the equilibrium measures.

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Contents

1 Introduction

Let (X, ω) be a compact Kähler manifold of dimension k. Our purpose in this paper is to develop a calculus on positive closed currents of bidegree (p, p) on X. We will also apply this calculus to prove some surprising uniqueness results for dynamical currents in their cohomology classes.

When S is a positive closed $(1, 1)$-current on X, it is possible to introduce its potential u satisfying the following equation with a normalization condition

$$dd^c u = S - \alpha \quad \text{and} \quad \int_X u\omega^k = 0,$$

where α is a smooth representative of the cohomology class of S. The function u is quasi-p.s.h.; it is defined everywhere and satisfies $dd^c u \geq -c\omega$ for some constant $c > 0$. This is the unique solution of the above equation and calculus problems on S can be transfered to computation on the potential u.

We have developed in [25] a theory of super-potentials associated to positive closed currents S of bidegree (p, p) in $\mathbb{P}^k$ (our approach can be easily extended to homogeneous manifolds). Let $\omega_{\rm FS}$ denote the Fubini-Study form on $\mathbb{P}^k$ normalized by $\int_{\mathbb{P}^k} \omega_{\rm FS}^k = 1$. Assume for simplicity that S is of mass 1, that is, S is cohomologous to $\omega_{\rm FS}^p$. One can always solve the equation

$$dd^c U_S = S - \omega_{\rm FS}^p \quad \text{and} \quad \langle U_S, \omega_{\rm FS}^{k-p+1} \rangle = 0. \tag{1.1}$$

But when $p > 1$, the current U_S is not unique and it is difficult to give U_S a value at every point, in order for example, to consider expressions like the wedge-product $U_S \wedge [V]$ where $[V]$ is a current associated to an analytic set V. In [25], we have introduced for S a super-potential $\mathcal{U}_S$ which is a function defined on the space of positive closed currents R of bidegree $(k-p+1, k-p+1)$ and of mass 1. More precisely, we have shown that it is possible to define[1]

$$\mathcal{U}_S(R) = \langle U_S, R \rangle := \limsup \langle U_S, R' \rangle$$

with R' smooth positive closed converging to R. The above formula is symmetric in R and S, that is, $\mathcal{U}_S(R) = \mathcal{U}_R(S)$. So, we also have $\mathcal{U}_S(R) = \langle S, U_R \rangle$ where

[1] in [25] we call $\mathcal{U}_S$ the super-potential of mean 0 of S.

U_R is a normalized solution of the equation $dd^c U_R = R - \omega_{\rm FS}^{k-p+1}$. In particular, $\mathcal{U}_S(R)$ does not depend on the choice of U_S and U_R.

The super-potentials appear as quasi-p.s.h. functions on an infinite dimensional space and the value $-\infty$ is admissible. The calculus we have obtained is satisfactory and permits to solve non trivial dynamical questions for holomorphic endomorphisms of $\mathbb{P}^k$ and polynomial automorphisms of $\mathbb{C}^k$. It also permits to give a useful intersection theory of positive closed currents in $\mathbb{P}^k$.

It will be important to extend such a calculus to arbitrary compact Kähler manifolds. There are however some important difficulties. First, according to Bost-Gillet-Soulé [5], if $p > 1$, it is not always possible to solve the equation (1.1) with U_S bounded from above. In some sense, using the potentials one may loose the positivity or the boundedness from below. Second, the approximation of arbitrary positive closed currents by smooth ones is only possible when a loss in positivity is allowed, see Theorem 2.4.4 below. The loss of positivity is under control but it is still a source of several technical difficulties. In general, the deformation of currents on non-homogeneous manifolds is a delicate problem.

In the present paper, we introduce the super-potentials of S as acting on the real vector space of closed currents R which are smooth and cohomologous to 0. Then $\mathcal{U}_S$ is defined as

$$\mathcal{U}_S(R) := \langle S, U_R \rangle,$$

where U_R is a smooth solution of $dd^c U_R = R$ satisfying some normalization conditions. This permits to develop the first steps of a theory of super-potential on an arbitrary compact Kähler manifold. In particular, we can define with some regularity assumption, the wedge-product $S_1 \wedge S_2$ where S_j are positive closed (p_j, p_j)-currents.

We then apply these notions to the dynamical study of automorphisms of a compact Kähler manifold. Let f be a holomorphic automorphism of X. The dynamical degree of order s of f is defined as the spectral radius of the pull-back operator f^* on the cohomology group $H^{s,s}(X, \mathbb{R})$. It follows from a result by Khovanskii-Teissier-Gromov [33] that the function $s \mapsto \log d_s$ is concave. In particular, we can assume that

$$1 = d_0 < d_1 < \cdots < d_p = \cdots = d_{p'} > \cdots > d_{k-1} > d_k = 1.$$

We have constructed in [20] for $1 \leq q \leq p$, Green (q, q)-currents T_q. In our context, they are the positive closed currents of bidegree (q, q) such that $f^*(T_q) = d_q T_q$, see Section 4.2 for the precise definition. Under the hypothesis that the dynamical degrees are distinct (i.e. $p = p'$), we also constructed and studied an ergodic invariant measure μ for f. The case of surfaces ($k = 2$) was studied by Cantat in [7]. Dynamically interesting automorphisms of surfaces are also constructed in Bedford-Kim [1] and McMullen [41].

Here, we propose a new approach using super-potentials to deal with convergence problems. We will show that the Green currents have Hölder continuous super-potentials. The following uniqueness result is quite surprising and can be applied to all q smaller than or equal to p. We refer to [29, 27, 26, 24, 25, 16] and the references therein for analogous results in other settings.

Theorem. *Let f be a holomorphic automorphism of a compact Kähler manifold X and d_s the dynamical degrees of f. Suppose V is a subspace of $H^{q,q}(X,\mathbb{R})$ invariant under f^*. Assume that all the (real and complex) eigenvalues of the restriction of f^* to V are of modulus strictly larger that d_{q-1}. Then each class in V contains at most one positive closed (q,q)-current.*

As a consequence, we deduce that given a positive closed (q,q)-current S, the convergence of the classes $(f^{n_i})^*[S]$, properly normalized, implies the convergence of the currents $(f^{n_i})^*(S)$, properly normalized. Here, $f^n := f \circ \cdots \circ f$ (n times) is the iterate of order n of f. The result applied to the current of integration on a subvariety Y of X gives a description of the asymptotic behavior of the inverse image of Y by f^n when $n \to \infty$. We also deduce that the Green currents are the unique positive closed currents in their cohomology classes without restricting to invariant currents.

Assume that the dynamical degrees of f are distinct, i.e. $p = p'$ (for surfaces this just means $d_1 > 1$). Assume also that the action of f^* on $H^{p,p}(X,\mathbb{R})$ satisfies the following condition which is always true for surfaces. Let H be the invariant subspace of $H^{p,p}(X,\mathbb{R})$ corresponding to eigenvalues of maximal modulus. Suppose that f^* restricted to H is diagonalizable over $\mathbb{C}$. This condition means that the Jordan form of f^* restricted to $H \otimes_{\mathbb{R}} \mathbb{C}$ is a diagonal matrix. Let T^+ be a Green (p,p)-current of f and T^- a Green $(k-p,k-p)$-current associated to f^{-1}. The hypothesis on $f^*_{|H}$ is a necessary and sufficient condition in order to have $T^+ \wedge T^- \neq 0$ for a suitable choice of T^+, T^-, see Proposition 4.4.1. These measures $T^+ \wedge T^-$ generate a real space $\mathcal{N}$ of finite dimension. We will show that the convex cone $\mathcal{N}^+$ of positive measures in $\mathcal{N}$ is closed, with a simplicial basis and that the measures μ on the extremal rays are ergodic. When the eigenvalues of $f^*_{|H}$ are all real positive, i.e. equal to d_p, μ is mixing.

We will show that any such measure μ is of maximal entropy $\log d_p$. Then, using a recent result of de Thélin [13] we deduce that μ is hyperbolic with precise estimates on the positive and the negative Lyapounov exponents. The Hölder continuity of the super-potentials of the Green currents implies that μ is moderate: if u belongs to a compact family of quasi-p.s.h. functions and $dd^c u \geq -\omega$ then $\langle \mu, e^{\lambda|u|}\rangle \leq c$ for some positive constants λ and c. As far as we know, this property is the strongest regularity property satisfied by the equilibrium measures in a quite general setting. It implies that any quasi-p.s.h. function is in $L^p(\mu)$ for all $1 \leq p < \infty$. Moreover, μ has no mass on proper analytic subsets of X. A result due to Katok [38, p.694] implies that the set of saddle periodic points is Zariski dense in X since its closure contains the support of μ.

We have tried to make the paper readable for non experts. In Section 2 we give background on positive closed currents and we introduce transforms on currents in a quite general context. We show how to regularize positive closed currents and how to solve the dd^c-equation in an arbitrary compact Kähler manifold. In Section 3 we construct an explicit structural variety for a given current R which is a difference of positive closed currents. So, R appears as the slice by $\{0\} \times X$ of a closed current $\mathcal{R}$ in $\mathbb{P}^1 \times X$. This is the main technical tool, which permits to use the powerful estimates on subharmonic functions in order to prove the convergence theorems. We also define here the intersection of currents. In the last section, we give the applications to the dynamics of automorphisms.

Main notations and conventions. Throughout the paper, except for some definitions in Section 2.2, (X, ω) is a compact Kähler manifold of dimension k. The notation $[V]$ or $[S]$ means the current of integration on an analytic set V or the class of a dd^c-closed (p,p)-current S in $H^{p,p}(X, \mathbb{C})$ or $H^{p,p}(X, \mathbb{R})$. Denote by $\pi : \widehat{X \times X} \to X \times X$ the blow-up along the diagonal Δ and $\widehat{\Delta} := \pi^{-1}(\Delta)$ the exceptional hypersurface in $\widehat{X \times X}$. The canonical projections of $X \times X$ on its factors are denoted by π_i and we define $\Pi_i := \pi_i \circ \pi$ for $i = 1, 2$. We also fix a Kähler form $\widehat{\omega}$ on $\widehat{X \times X}$. Let $\mathcal{C}_p$ denote the convex cone of positive closed (p,p)-currents on X, $\mathcal{D}_p$ the real space generated by $\mathcal{C}_p$ and $\mathcal{D}_p^0$ the subspace of currents in $\mathcal{D}_p$ which belong to the class 0 in $H^{p,p}(X, \mathbb{R})$. We consider on these spaces the norms $\|\cdot\|_{\mathcal{C}^{-l}}$, $\|\cdot\|_*$ and the $*$-topology defined in Section 2.2. On $\mathcal{C}_p$ or on $*$-bounded (i.e. bounded with respect to $\|\cdot\|_*$) subsets of $\mathcal{D}_p$, the $*$-topology coincides with the weak topology on currents. The current Θ_0, its deformations Θ_θ with $\theta \in \mathbb{P}^1$, the associated transforms $\mathcal{L}_0$, $\mathcal{L}_\theta$ and the transform $\mathcal{L}_K$ are defined in Section 2.4. The deformations $S_\theta := \mathcal{L}_\theta(S)$ of a current S and the associated structural line $(S_\theta)_{\theta \in \mathbb{P}^1}$ are introduced in Sections 2.4 and 3.1. The super-potential of a current S in $\mathcal{D}_p$, normalized by a fixed family α of closed (p,p)-forms, is denoted by $\mathcal{U}_S$. If S belongs to $\mathcal{D}_p^0$, then $\mathcal{U}_S$ does not depend on the choice of α, see Section 3.2. Finally, most of the constants depend only on (X, ω). The notations $\gtrsim$, $\lesssim$ mean inequalities up to a multiplicative constant and we will write $\sim$ when both inequalities are satisfied.

2 Background on positive closed currents

In this section, we recall some basic facts on Hodge theory for compact Kähler manifolds, some properties of positive closed currents and plurisubharmonic functions. We refer to [10, 11, 28, 37, 40, 46] for more detailed expositions on these subjects.

2.1 Compact Kähler manifolds

• **Hodge cohomology groups.** Consider a compact Kähler manifold X of dimension k. Let $H^r(X, \mathbb{R})$ and $H^r(X, \mathbb{C})$ denote the de Rham cohomology groups of real and complex smooth r-forms. Let $H^{p,q}(X,\mathbb{C})$, $p+q=r$, be the subspace of $H^r(X,\mathbb{C})$ generated by the classes of closed (p,q)-forms. The Hodge theory asserts that

$$H^r(X,\mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(X,\mathbb{C}) \quad \text{and} \quad H^{p,q}(X,\mathbb{C}) = \overline{H^{q,p}(X,\mathbb{C})}.$$

For $p=q$, define

$$H^{p,p}(X,\mathbb{R}) := H^{p,p}(X,\mathbb{C}) \cap H^{2p}(X,\mathbb{R}),$$

then

$$H^{p,p}(X,\mathbb{C}) = H^{p,p}(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

The cup-product $\smile$ on $H^{p,p}(X,\mathbb{R}) \times H^{k-p,k-p}(X,\mathbb{R})$ is defined by

$$([\beta],[\beta']) \mapsto [\beta] \smile [\beta'] := \int_X \beta \wedge \beta'$$

where β and β' are smooth closed forms. The last integral depends only on the classes of β and β'. The bilinear form $\smile$ is non-degenerate and induces a duality (Poincaré duality) between $H^{p,p}(X,\mathbb{R})$ and $H^{k-p,k-p}(X,\mathbb{R})$. In the definition of $\smile$ one can take β' smooth and β a current in the sense of de Rham. So, $H^{p,p}(X,\mathbb{R})$ can be defined as the quotient of the space of real closed (p,p)-currents by the subspace of d-exact currents. Recall that a (p,p)-current β is real if $\overline{\beta} = \beta$. When β is a real (p,p)-current such that $dd^c\beta = 0$, by the dd^c-lemma [10, 46], the integral $\int_X \beta \wedge \beta'$ is also independent of the choice of β' smooth and closed in a fixed cohomology class. So, using the duality, one can associate to β a class in $H^{p,p}(X,\mathbb{R})$.

• **Blow-up along the diagonal.** The integration on the diagonal Δ of $X \times X$ defines a real closed (k,k)-current $[\Delta]$ which is positive, see Section 2.2 for the notion of positivity. By Künneth formula, we have a canonical isomorphism

$$H^{k,k}(X \times X,\mathbb{C}) \simeq \sum_{0 \leq r \leq k} H^{r,k-r}(X,\mathbb{C}) \otimes H^{k-r,r}(X,\mathbb{C}).$$

Hence, $[\Delta]$ is cohomologous to a smooth real closed (k,k)-form α_Δ which is a finite combination of forms of type $\beta(x) \wedge \beta'(y)$. Here, β and β' are closed forms on X of bidegree $(r,k-r)$ and $(k-r,r)$ respectively, and (x,y) denotes the coordinates of $X \times X$. In other words, if π_i denote the projections of $X \times X$ on its factors, then α_Δ is a combination of $\pi_1^*(\beta) \wedge \pi_2^*(\beta')$. So, α_Δ satisfies $d_x\alpha_\Delta = d_y\alpha_\Delta = 0$. Replacing $\alpha_\Delta(x,y)$ by $\frac{1}{2}\alpha_\Delta(x,y) + \frac{1}{2}\alpha_\Delta(y,x)$ allows to assume that α_Δ is *symmetric*, i.e. invariant by the involution $(x,y) \mapsto (y,x)$.

Let $\pi : \widehat{X \times X} \to X \times X$ be the blow-up of $X \times X$ along Δ and $\widehat{\Delta} := \pi^{-1}(\Delta)$ the exceptional hypersurface. By a theorem of Blanchard [4], $\widehat{X \times X}$ is a compact

Kähler manifold. We fix a Kähler form $\widehat{\omega}$ on $\widehat{X \times X}$. According to Gillet-Soulé [32, 1.3.6], there is a real smooth closed $(k-1, k-1)$-form η on $\widehat{X \times X}$ such that $\pi^*(\alpha_\Delta)$ is cohomologous to $[\widehat{\Delta}] \wedge \eta$, where $[\widehat{\Delta}]$ is the positive closed $(1,1)$-current of integration on $\widehat{\Delta}$. Hence, $\pi_*([\widehat{\Delta}] \wedge \eta)$ is cohomologous to α_Δ and to $[\Delta]$. On the other hand, $\pi_*([\widehat{\Delta}] \wedge \eta)$ is supported on Δ and is equal to a product of $[\Delta]$ by a function. We deduce that $\pi_*([\widehat{\Delta}] \wedge \eta) = [\Delta]$. The map $(x, y) \mapsto (y, x)$ induces an involution on $\widehat{X \times X}$. We can also choose η symmetric with respect to this involution.

Let γ be a real closed $(1,1)$-form on $\widehat{X \times X}$, cohomologous to $[\widehat{\Delta}]$. We can choose γ symmetric. We will see later that there is a quasi-p.s.h. function φ on $\widehat{X \times X}$ such that $dd^c\varphi = [\widehat{\Delta}] - \gamma$. This function is necessarily symmetric. Subtracting from φ a constant allows to assume that $\varphi < -2$.

• **Local coordinates near Δ and $\widehat{\Delta}$.** Consider a local coordinate system $x = (x_1, \ldots, x_k)$ on a chart of X. For simplicity assume that the ball W of center 0 and of radius 1 is strictly contained in this chart. For the neighbourhood $W \times W$ of $(0,0)$ in $X \times X$, we will use the coordinates $(x, y) = (x_1, \ldots, x_k, y_1, \ldots, y_k)$. The diagonal Δ contains the point $(0,0)$ and is given by the equation $x = y$. Define $x' := x - y$. Then (x', y) is also a coordinate system of $W \times W$ and Δ is given by $x' = 0$. Consider the submanifold M of $\mathbb{C}^k \times \mathbb{C}^k \times \mathbb{P}^{k-1}$ defined by

$$M := \big\{(x', y, [v]) \in \mathbb{C}^k \times \mathbb{C}^k \times \mathbb{P}^{k-1}, \quad x' \in [v]\big\},$$

where $[v] = [v_1 : \cdots : v_k]$ denotes the homogeneous coordinates of $\mathbb{P}^{k-1}$. Recall that x' belongs to $[v]$ if and only if x' and v are proportional. The submanifold M is the blow-up of $\mathbb{C}^k \times \mathbb{C}^k$ along $x' = 0$. So, we identify $\pi^{-1}(W \times W)$ with an open set in M defined by $\|x' + y\| = \|x\| < 1$ and $\|y\| < 1$.

Consider a point $(a, b, [u])$ in $\widehat{\Delta}$. We have necessarily $a = 0$. For simplicity, assume that the first coordinate of u is the largest one. Therefore, we can write $[u] = [1 : u_2 : \cdots : u_k]$ with $|u_i| \leq 1$. In a neighbourhood of $(0, b, [u])$, the first coordinate of v does not vanish and we can write $[v] = [1 : v_2 : \cdots : v_k]$ with $|v_i| < 2$. Write $v' := (v_2, \ldots, v_k)$. Then, (x_1', y, v') is a local coordinate system for a neighbourhood of $(0, b, [u])$. Here, $\widehat{\Delta}$ is given by the equation $x_1' = 0$. We also have

$$\pi(x_1', y, v') = (x', y) = (x_1', x_1'v', y) \quad \text{and} \quad \Pi_2(x_1', y, v') = y.$$

We see that Π_2 and its restriction to $\widehat{\Delta}$ are submersions. In the same way, we prove that Π_1 and its restriction to $\widehat{\Delta}$ are also submersions.

2.2 Positive currents and plurisubharmonic functions

• **Positive closed currents.** A smooth (p, p)-form ϕ on a general complex manifold of dimension k is *positive* if it can be written in local charts as a finite combination with positive coefficients of forms of type

$$(i\alpha_1 \wedge \overline{\alpha}_1) \wedge \ldots \wedge (i\alpha_p \wedge \overline{\alpha}_p)$$

where α_i are $(1,0)$-forms. The positivity is a pointwise property and does not depend on local coordinates. A (p,p)-current S is *weakly positive* if $S \wedge \phi$ is a positive measure for every smooth positive $(k-p, k-p)$-form ϕ. The current S is *positive* if $S \wedge \phi$ is a positive measure for every smooth weakly positive $(k-p, k-p)$-form ϕ. The notions of positivity and weak positivity coincide for $p = 0, 1, k-1$ and k. We say that S is *negative* if $-S$ is positive and we write $S \geq S'$, $S' \leq S$ when $S - S'$ is positive. Note that positive and negative currents are real. If S, S' are positive and S' is smooth then $S \wedge S'$ is positive. Let V be an analytic subset of pure codimension p. Then, the integration on the regular part of V defines a positive closed (p,p)-current that we denote by $[V]$. A (p,p)-current S is said to be *strictly positive* if in local coordinates x, we have $S \geq \epsilon (dd^c \|x\|^2)^p$ for some $\epsilon > 0$.

Let (X, ω) be a compact Kähler manifold of dimension k. If S is a positive or negative (p,p)-current on X, define *the mass*[2] of S by

$$\|S\| := |\langle S, \omega^{k-p} \rangle|.$$

Let $\mathcal{C}_p$ denote the cone of positive closed (p,p)-currents on X, $\mathcal{D}_p$ the real space generated by $\mathcal{C}_p$ and $\mathcal{D}_p^0$ the space of currents $S \in \mathcal{D}_p$ such that $[S] = 0$ in $H^{p,p}(X, \mathbb{R})$. The duality between the cohomology groups implies that if S is a current in $\mathcal{C}_p$, its mass depends only on the class $[S]$ in $H^{p,p}(X, \mathbb{R})$. Define *the norm* $\|\cdot\|_*$ on $\mathcal{D}_p$ by

$$\|S\|_* := \min \|S^+\| + \|S^-\|,$$

where the minimum is taken over S^+, S^- in $\mathcal{C}_p$ such that $S = S^+ - S^-$. A subset in $\mathcal{D}_p$ is $*$-*bounded* if it is bounded with respect to the $\|\cdot\|_*$-norm. We will consider on $\mathcal{D}_p$ and $\mathcal{D}_p^0$ the following $*$-*topology*. We say that S_n *converge to* S *in* $\mathcal{D}_p$ if $S_n \to S$ weakly and if $\|S_n\|_*$ is bounded by a constant independent of n. Note that the $*$-topology restricted to $\mathcal{C}_p$ or to a $*$-bounded subset of $\mathcal{D}_p$ coincides with the weak topology. We will see in Theorem 2.4.4 below that smooth forms are dense in $\mathcal{D}_p$ and $\mathcal{D}_p^0$ for the $*$-topology.

Consider some natural *norms* and *distances* on $\mathcal{D}_p$. For $l \geq 0$, let $[l]$ denote the integer part of l. Let $\mathcal{C}^l_{p,q}$ be the space of (p,q)-forms whose coefficients admit all derivatives of order $\leq [l]$ and these derivatives are $(l-[l])$-Hölder continuous. We use here the sum of $\mathcal{C}^l$-norms of the coefficients for a fixed atlas. If S and S' are currents in $\mathcal{D}_p$, define[3]

$$\|S\|_{\mathcal{C}^{-l}} := \sup_{\|\Phi\|_{\mathcal{C}^l} \leq 1} |\langle S, \Phi \rangle| \quad \text{and} \quad \mathrm{dist}_l(S, S') := \|S - S'\|_{\mathcal{C}^{-l}}$$

[2]in this case, this quantity is equivalent to the mass norm for currents of order 0, see [28].

[3]the definition is meaningful for any current S of order 0 and $\|\cdot\|_{\mathcal{C}^{-0}}$ is equivalent to the mass norm in the usual sense, see [28].

where Φ is a test smooth $(k-p,k-p)$-form on X. Observe that $\|\cdot\|_{\mathcal{C}^{-l}} \lesssim \|\cdot\|_*$ for every $l \geq 0$. The following result is proved as in [25] using the theory of interpolation between Banach spaces.

Proposition 2.2.1. *Let l and l' be real strictly positive numbers with $l < l'$. Then on any $*$-bounded subset of $\mathcal{D}_p$, the topology induced by dist_l or by $\mathrm{dist}_{l'}$ coincides with the weak topology. Moreover, on any $*$-bounded subset of $\mathcal{D}_p$, there is a constant $c_{l,l'} > 0$ such that*

$$\mathrm{dist}_{l'} \leq \mathrm{dist}_l \leq c_{l,l'}[\mathrm{dist}_{l'}]^{l/l'}.$$

In particular, a function on a $*$-bounded subset of $\mathcal{D}_p$ is Hölder continuous with respect to dist_l if and only if it is Hölder continuous with respect to $\mathrm{dist}_{l'}$.

• **Plurisubharmonic functions.** Consider a general (connected) complex manifold X. An upper semi-continuous function $u : X \to \mathbb{R} \cup \{-\infty\}$, not identically $-\infty$, is *plurisubharmonic* (p.s.h. for short) if its restriction to each holomorphic disc in X is subharmonic or identically equal to $-\infty$. If u is a p.s.h. function then u belongs to L^p_{loc} for $1 \leq p < \infty$, and $dd^c u$ is a positive closed $(1,1)$-current on X. Conversely, if S is a positive closed $(1,1)$-current, it can be locally written as $S = dd^c u$ with u p.s.h. A subset of X is *locally pluripolar* if it is locally contained in the pole set $\{u = -\infty\}$ of a p.s.h. function. P.s.h. functions satisfy a maximum principle. In particular, on a compact manifold, p.s.h. functions are constant. A function u on X is *quasi-p.s.h.* if it is locally a difference of a p.s.h. function and a smooth function.

Assume now that X is a compact Kähler manifold of dimension k and ω is a Kähler form on X. If u is a quasi-p.s.h. function on X then $dd^c u + c\omega$ is a positive closed $(1,1)$-current for $c > 0$ large enough. Conversely, if S is a positive closed $(1,1)$-current and α is a real closed smooth $(1,1)$-form cohomologous to S, then there is a quasi-p.s.h. function u such that $dd^c u = S - \alpha$. The function u is unique up to an additive constant. A subset of X is *pluripolar* if it is contained in the pole set $\{u = -\infty\}$ of a quasi-p.s.h. function u.

A function is called *d.s.h.* if it is equal outside a pluripolar set to a difference of two quasi-p.s.h. functions. We identify two d.s.h. functions if they are equal out of a pluripolar set. If u is d.s.h. then $dd^c u$ is a difference of two positive closed $(1,1)$-currents which are cohomologous. Conversely, if S^+ and S^- are positive closed $(1,1)$-currents in the same cohomology class then $S^+ - S^- = dd^c u$ for some d.s.h. function u. The function u is unique up to an additive constant. There are several equivalent norms on the space of d.s.h. functions. We consider the following one, see [21]

$$\|u\|_{\mathrm{DSH}} := \|u\|_{L^1} + \|dd^c u\|_*.$$

We have the following proposition [21].

Proposition 2.2.2. *Let u be a d.s.h. function on X. Then there exist two quasi-p.s.h. functions u^+, u^- such that*

$$u = u^+ - u^-, \quad \|u^\pm\|_{L^1} \leq c\|u\|_{\mathrm{DSH}}, \quad \text{and} \quad dd^c u^\pm \geq -c\|u\|_{\mathrm{DSH}}\omega,$$

where $c > 0$ is a constant independent of u.

We deduce from this proposition and the fundamental exponential estimate for p.s.h. functions [37] the following result, see also [17].

Proposition 2.2.3. *There are constants $\lambda > 0$ and $c > 0$ such that if u is a d.s.h. function with $\|u\|_{\mathrm{DSH}} \leq 1$ then*

$$\int_X e^{\lambda|u|}\omega^k \leq c.$$

We will need the following version of the exponential estimate for d.s.h. functions on $\mathbb{P}^1$ and for ω_{FS} the Fubini-Study form on $\mathbb{P}^1$.

Lemma 2.2.4. *Let u be a d.s.h. function on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Assume that u vanishes outside the unit disc of $\mathbb{C}$ and that $dd^c u$ is a measure of mass at most equal to 1. Then there are constants $\lambda > 0$ and $c > 0$ independent of u such that*

$$\int_{\mathbb{P}^1} e^{\lambda|u|}\omega_{\mathrm{FS}} \leq c,$$

In particular, if B is a disc of radius r, $0 < r < 1/2$, then $\inf_B |u| \leq -c' \log|r|$ for some constant $c' > 0$ independent of u, B and r.

Proof. Write $dd^c u = \nu^+ - \nu^-$ where $\nu^\pm$ are probability measures with support in the unit disc. Define for $z \in \mathbb{C}$

$$u^\pm(z) := \int_{\mathbb{C}} \log|z - \xi| d\nu^\pm(\xi).$$

Observe that $\|u^\pm\|_{L^1(\mathbb{P}^1)}$ are bounded by a constant independent of $\nu^\pm$. We also have

$$\lim_{z\to\infty} u^\pm(z) - \log|z| = 0 \quad \text{and} \quad dd^c u^\pm = \nu^\pm - \delta_\infty$$

where δ_∞ is the Dirac mass at ∞. It follows that

$$\lim_{z\to\infty} u^+(z) - u^-(z) = 0 \quad \text{and} \quad dd^c(u^+ - u^-) = \nu^+ - \nu^- = dd^c u.$$

So, $u^+ - u^-$ and u differe by a constant. The fact that u is supported in the unit disc implies that $u = u^+ - u^-$. We deduce that $\|u\|_{L^1}$ is bounded by a constant independent of u, and then $\|u\|_{\mathrm{DSH}}$ is bounded by a constant independent of u. Proposition 2.2.3 implies the result. □

• **Slicing of positive closed current.** Let V be a complex manifold of dimension l. We are interested in families of currents parametrized by V which are slices of some closed current $\mathcal{R}$ in $V \times X$. Let π_V and π_X denote the canonical projections from $V \times X$ on its factors. We have the following proposition where currents on $\{\theta\} \times X$ are identified with currents on X, see also [22].

Proposition 2.2.5. *Let $\mathcal{R}$ be a positive closed (s,s)-current in $V \times X$ with $s \leq k$. Then there is a locally pluripolar subset E of V such that the slice $\langle \mathcal{R}, \pi_V, \theta\rangle$ exists for $\theta \in V \setminus E$. Moreover, $\langle \mathcal{R}, \pi_V, \theta\rangle$ is a positive closed (s,s)-current on $\{\theta\} \times X$ and its class in $H^{s,s}(X, \mathbb{R})$ does not depend on θ.*

Recall that slicing is the generalization of restriction of forms to level sets of holomorphic maps. It can be viewed as a version of Fubini's theorem or Sard's theorem for currents. The operation is well-defined for currents $\mathcal{R}$ of order 0 and of bidegree $\leq (k,k)$ such that $\partial\mathcal{R}$ and $\overline{\partial}\mathcal{R}$ are of order 0. When $\mathcal{R}$ is a smooth form, $\langle \mathcal{R}, \pi_V, \theta\rangle$ is simply the restriction of $\mathcal{R}$ to $\{\theta\} \times X$. When $\mathcal{R}$ is the current of integration on an analytic subset Y of $V \times X$, $\langle \mathcal{R}, \pi_V, \theta\rangle$ is the current of integration on the analytic set $Y \cap \{\theta\} \times X$ for θ generic.

In general, if ϕ is a smooth form on $V \times X$ then $\langle \mathcal{R} \wedge \phi, \pi_V, \theta\rangle = \langle \mathcal{R}, \pi_V, \theta\rangle \wedge \phi$. Slicing commutes with the operations ∂ and $\overline{\partial}$. So, in our situation, since $\mathcal{R}$ is closed, $\langle \mathcal{R}, \pi_V, \theta\rangle$ is also closed. The following description shows that $\langle \mathcal{R}, \pi_V, \theta\rangle$ is positive.

Let z denote the coordinates in a chart of V and λ_V the standard volume form. Let $\psi(z)$ be a positive smooth function with compact support such that $\int \psi\lambda_V = 1$. Define $\psi_\epsilon(z) := \epsilon^{-2l}\psi(\epsilon^{-1}z)$ and $\psi_{\theta,\epsilon}(z) := \psi_\epsilon(z-\theta)$. The measures $\psi_{\theta,\epsilon}\lambda_V$ approximate the Dirac mass at θ. For every smooth test form Ψ of bidegree $(k-s, k-s)$ on $V \times X$ one has

$$\langle \mathcal{R}, \pi_V, \theta\rangle(\Psi) = \lim_{\epsilon\to 0}\langle \mathcal{R} \wedge \pi_V^*(\psi_{\theta,\epsilon}\lambda_V), \Psi\rangle \tag{2.1}$$

when $\langle \mathcal{R}, \pi_V, \theta\rangle$ exists. This property holds for all choice of ψ. Conversely, when the previous limit exists and is independent of ψ, it defines the current $\langle \mathcal{R}, \pi_V, \theta\rangle$ and one says that $\langle \mathcal{R}, \pi_V, \theta\rangle$ *is well-defined.* The following formula holds for smooth forms Ω of maximal degree with compact support in V:

$$\int_{\theta\in V} \langle \mathcal{R}, \pi_V, \theta\rangle(\Psi)\Omega(\theta) \quad = \quad \langle \mathcal{R} \wedge \pi_V^*(\Omega), \Psi\rangle. \tag{2.2}$$

Proof of Proposition 2.2.5. Since the problem is local on V, we can assume that V is a ball in $\mathbb{C}^l$ and z are the standard coordinates. It is enough to consider real test forms Ψ with compact suppport. Define $\phi := (\pi_V)_*(\mathcal{R} \wedge \Psi)$. This is a current of bidegree $(0,0)$ on V. Observe that $dd^c\Psi$ can be written as a difference of positive closed forms on $V \times X$, not necessarily with compact support. It follows that $dd^c\phi = (\pi_V)_*(\mathcal{R} \wedge dd^c\Psi)$ is a difference of positive closed currents.

Therefore, ϕ can be considered as a d.s.h. function. We have

$$\langle \mathcal{R} \wedge \pi_V^*(\psi_{\theta,\epsilon}\lambda_V), \Psi\rangle = \int_V \phi\psi_{\theta,\epsilon}\lambda_V.$$

Classical properties of p.s.h. functions imply that for θ outside a pluripolar set (because ϕ is only d.s.h.) the last integral converges to $\phi(\theta)$ when $\epsilon \to 0$. So, for such a θ, the limit in (2.1) exists and does not depend on ψ.

Choose a pluripolar set $E \subset V$ such that the previous convergence holds for θ outside E and for a countable family $\mathcal{F}$ of test forms Ψ. We choose a family $\mathcal{F}$ which is dense for the $\mathcal{C}^0$-topology. The density implies that we have the convergence for a test form Ψ strictly positive near $\{\theta\} \times X$. This, the density of $\mathcal{F}$ together with the positivity of $\mathcal{R}$ imply the convergence for every Ψ. Hence, $\langle \mathcal{R}, \pi_V, \theta\rangle$ is well-defined for $\theta \notin E$ and is a positive closed current on $\{\theta\} \times X$.

We have $\langle \mathcal{R}, \pi_V, \theta\rangle(\Psi) = \phi(\theta)$. Consider a closed $(k-s, k-s)$-form Φ on X and $\Psi := \pi_X^*(\Phi)$. If ϕ is defined as above, we have $d\phi = 0$. Therefore, ϕ is a constant function and $\langle \mathcal{R}, \pi_V, \theta\rangle(\Psi)$ does not depend on θ. If $\langle \mathcal{R}, \pi_V, \theta\rangle$ is identified with a current on X, then $\langle \mathcal{R}, \pi_V, \theta\rangle(\Phi)$ is independent of θ. This, together with Poincaré's duality, implies that the class of $\langle \mathcal{R}, \pi_V, \theta\rangle$ in $H^{s,s}(X, \mathbb{R})$ does not depend on θ. □

Remark 2.2.6. Assume that $\langle \mathcal{R}, \pi_V, \theta\rangle$ is defined for θ outside a set of zero measure and that $\theta \mapsto \langle \mathcal{R}, \pi_V, \theta\rangle$ can be extended to a continuous map with values in the space of currents of order 0. Then, by definition of slicing, (2.1) and (2.2), $\langle \mathcal{R}, \pi_V, \theta\rangle$ is defined for every θ and coincides with the continuous extension of $\theta \mapsto \langle \mathcal{R}, \pi_V, \theta\rangle$. If $\mathcal{R}_n$ are positive closed currents converging to $\mathcal{R}$, we can prove that there is a subsequence $\mathcal{R}_{n_i}$ with $\langle \mathcal{R}_{n_i}, \pi_V, \theta\rangle \to \langle \mathcal{R}, \pi_V, \theta\rangle$ for almost every θ. Indeed, for a bounded sequence of d.s.h. functions on V we can extract a subsequence which converges almost everywhere.

2.3 Transforms on currents

• **General transforms on currents.** We recall here a general idea how to construct linear operators on currents which are useful in geometrical questions. Let X_1, X_2 and Z be Riemannian manifolds and τ_1, τ_2 smooth maps from Z to X_1 and X_2. Let Θ be a fixed current on Z. Define for a current S on X_1 another current $\mathcal{L}_\Theta(S)$ on X_2 by

$$\mathcal{L}_\Theta(S) := (\tau_2)_*\big(\tau_1^*(S) \wedge \Theta\big)$$

when the last expression is meaningful. The current $\tau_1^*(S)$ is well-defined if S is a bounded form or if τ_1 is a submersion. The operator $(\tau_2)_*$ is well-defined if τ_2 is proper, in particular, when Z is compact. Assume that Z is compact. Then, $\mathcal{L}_\Theta$ is well-defined on smooth currents S. Let Θ' denote the push-forward of Θ to $X_1 \times X_2$ by the map (τ_1, τ_2). Then, Θ' defines a transform $\mathcal{L}_{\Theta'}$ where Z is replaced by $X_1 \times X_2$. The transform $\mathcal{L}_{\Theta'}$ is equal to $\mathcal{L}_\Theta$ on smooth forms S and this useful property may be extended to larger spaces of currents.

In this paper, we consider the following situation used in Gillet-Soulé [32] and [5, 19, 20, 45], see also [3, 36, 42, 21]. We use the notations introduced in Section 2.1. Consider a current Θ of bidegree (r, s) on $\widehat{X \times X}$. If S is a current on X, define the transform $\mathcal{L}_\Theta(S)$ of S by

$$\mathcal{L}_\Theta(S) := (\Pi_2)_*\big(\Pi_1^*(S) \wedge \Theta\big).$$

This definition makes sense if the last wedge-product is well-defined, in particular when Θ or S is smooth.

If S is of bidegree (p, q) then $\mathcal{L}_\Theta(S)$ is of bidegree $(p + r - k, q + s - k)$. So, we say that the transform $\mathcal{L}_\Theta$ is *of bidegree* $(r - k, s - k)$. The bidegree may be negative. In what follows, we will be interested in the cases where $r = s = k$ or $r = s = k - 1$, and S is a current in $\mathcal{D}_p$. The current Θ will be real and smooth or smooth outside $\widehat{\Delta}$. If Θ is positive or negative, we say that the transform $\mathcal{L}_\Theta$ is *positive* or *negative* respectively.

Example 2.3.1. Consider $\Theta_0 := [\widehat{\Delta}] \wedge \eta$ where η is the smooth real closed form of bidegree $(k - 1, k - 1)$ chosen in Section 2.1, and define $\mathcal{L}_0 := \mathcal{L}_{\Theta_0}$. Since Π_1 and its restriction to $\widehat{\Delta}$ are submersions, $\mathcal{L}_0$ can be extended continuously to any current S. We have $\pi_*(\Theta_0) = [\Delta]$. So, if S is a smooth form, then

$$\mathcal{L}_0(S) = (\pi_2)_*\big(\pi_1^*(S) \wedge [\Delta]\big) = S.$$

By continuity, $\mathcal{L}_0$ is equal to the identity on all currents S. If S is in $\mathcal{D}_p$, using the theory of intersection with positive closed $(1, 1)$-currents [8, 9, 31] and that $[\widehat{\Delta}] = dd^c\varphi + \gamma$, we obtain for S in $\mathcal{D}_p$ that if $S' := \Pi_1^*(S)$

$$S = \mathcal{L}_0(S) = (\Pi_2)_*\big(dd^c(\varphi S' \wedge \eta) + \gamma \wedge S' \wedge \eta\big),$$

see also [19]. We will construct some deformations $\mathcal{L}_\theta$ of $\mathcal{L}_0$, i.e. transforms associated to some deformations Θ_θ of Θ_0.

• **Regular and semi-regular transforms.** Consider now a situation used in [19, 20]. Let Θ be a form which is smooth outside $\widehat{\Delta}$ and such that

$$|\Theta| \lesssim -\log \mathrm{dist}(\cdot, \widehat{\Delta}) \quad \text{and} \quad |\nabla\Theta| \lesssim \mathrm{dist}(\cdot, \widehat{\Delta})^{-1}$$

near $\widehat{\Delta}$. Here, the estimate on $\nabla\Theta$ means an estimate on the gradients of the coefficients of Θ for a fixed atlas. Transforms associated to such forms Θ are called *semi-regular* (when Θ is smooth, we say that $\mathcal{L}_\Theta$ is *regular*). The form $\Theta' := \pi_*(\Theta)$ is smooth outside Δ. Using the local coordinates described in Section 2.1, one proves that

$$|\Theta'| \lesssim -\log \mathrm{dist}(\cdot, \Delta)\mathrm{dist}(\cdot, \Delta)^{2-2k}$$

and

$$|\nabla\Theta'| \lesssim \mathrm{dist}(\cdot,\Delta)^{1-2k}$$

near Δ, see [19]. In particular, the coefficients of Θ' restricted to $X \times \{y\}$ are in $L^{1+1/k}$ for every $y \in X$.

Recall that $\mathcal{L}_\Theta$ is defined for S smooth. Since $\Pi_1^*(S) \wedge \Theta$ has no mass on $\widehat{\Delta}$, we have

$$\mathcal{L}_\Theta(S) = (\pi_2)_*\big(\pi_1^*(S) \wedge \Theta'\big).$$

The wedge-product $\pi_1^*(S) \wedge \Theta'$ has no mass on Δ. So, one has to integrate only outside Δ. Then, using the estimates on $|\Theta'|$, $|\nabla\Theta'|$ and the Hölder inequality, we obtain the following result, see [19, 20].

Proposition 2.3.2. *Any semi-regular transform can be extended to a linear continuous operator from the space of currents of order* 0 *to the space of* $L^{1+1/k}$ *forms. It defines a linear continuous operator from the space of* L^q *forms,* $q \geq 1$*, to the space of* $L^{q'}$ *forms where* q' *is given by* $q'^{-1} + 1 = q^{-1} + [1+1/k]^{-1}$ *if* $q < k+1$ *and* $q' = \infty$ *if* $q \geq k+1$*. It also defines a linear continuous operator from the space of* L^∞ *forms to the space of* $\mathcal{C}^1$ *forms.*

The following result is a direct consequence of Proposition 2.3.2.

Corollary 2.3.3. *Let* $\mathcal{L}_1, \ldots, \mathcal{L}_{k+2}$ *be semi-regular transforms of bidegree* $(0,0)$*. If* S *is a current of order* 0*, then* $S' := \mathcal{L}_{k+2} \circ \cdots \circ \mathcal{L}_1(S)$ *is a form of class* $\mathcal{C}^1$*. Moreover, we have* $\|S'\|_{\mathcal{C}^1} \leq c\|S\|$*, where* $c > 0$ *is a constant independent of* S*.*

We will need the following lemma.

Lemma 2.3.4. *Assume that* Θ *is a smooth positive closed* (k,k)*-form. Then* $\mathcal{L}_\Theta$ *defines a linear map from* $\mathcal{D}_p$ *to itself which preserves* $\mathcal{C}_p$*,* $\mathcal{D}_p^0$ *and is continuous with respect to the* $*$*-topology. Moreover, if* S *is in* $\mathcal{D}_p$*, then* $\|\mathcal{L}_\Theta(S)\|_* \leq c\|\Theta\|\|S\|_*$ *for some constant* $c > 0$ *independent of* Θ *and* S*.*

Proof. Since Π_1 is a submersion, by definition, $\mathcal{L}_\Theta$ is a linear continuous map on currents. It is clear that $\mathcal{L}_\Theta$ preserves $\mathcal{C}_p$, $\mathcal{D}_p$ and $\mathcal{D}_p^0$. We only have to prove the estimate on $\|\mathcal{L}_\Theta(S)\|_*$. We can assume that S is positive. Since Π_1 is a submersion, we have $\|\Pi_1^*(S)\| \lesssim \|S\|$. Recall that the mass of a positive closed current can be computed cohomologically. Therefore,

$$\|\Pi_1^*(S) \wedge \Theta\| \lesssim \|\Theta\|\|\Pi_1^*(S)\| \lesssim \|\Theta\|\|S\|.$$

The continuity of $(\Pi_2)_*$ implies the result. □

• **Symmetric transforms.** The map $(x,y) \mapsto (y,x)$ on $X \times X$ induces an involution on $\widehat{X \times X}$. In order to simplify notations, we will only consider the transforms $\mathcal{L}_\Theta$ associated to forms Θ which are invariant by this involution. We say that $\mathcal{L}_\Theta$ is *symmetric*. Let Ψ be a smooth test form on X of the apropriate bidegree. If S is smooth then we deduce from the symmetry of $\mathcal{L}_\Theta$ that

$$\langle \mathcal{L}_\Theta(S), \Psi\rangle = \int_{\widehat{X\times X}} \Pi_1^*(S)\wedge\Theta\wedge\Pi_2^*(\Psi) = \langle S, (\Pi_1)_*(\Pi_2^*(\Psi)\wedge\Theta)\rangle = \langle S, \mathcal{L}_\Theta(\Psi)\rangle.$$

When S is closed and Φ is a smooth test form of the apropriate bidegree, we have

$$\langle \mathcal{L}_\Theta(S), dd^c\Phi\rangle = \int_{\widehat{X\times X}} \Pi_1^*(S)\wedge dd^c\Theta\wedge\Pi_2^*(\Phi) = \langle \mathcal{L}_{dd^c\Theta}(S), \Phi\rangle.$$

Observe that the smoothness of S is superflous when Θ is smooth. The previous identities may be extended to some cases where S and Θ are not smooth using a regularization on S.

2.4 Regularization and Green potential

• **Deformation of the identity transform.** We introduce in this section a family of regular transforms $\mathcal{L}_\theta$, $\theta\in\mathbb{P}^1\setminus\{0\}$, of bidegree $(0,0)$ which is a continuous deformation of the identity transform $\mathcal{L}_0$ considered in Example 2.3.1.

We use the notations of Section 2.1. Consider the following regularization of the function φ. Recall that $\varphi\leq -2$. Let χ be a smooth convex increasing function on $\mathbb{R}\cup\{-\infty\}$ such that $\chi(t)=t$ for $t\geq 0$, $\chi(t)=-1$ for $t\leq -2$ and $0\leq\chi'\leq 1$. Define

$$\chi_\theta(t) := \chi(t-\log|\theta|)+\log|\theta| \quad \text{and} \quad \varphi_\theta := \chi_\theta(\varphi).$$

When $|\theta|$ decreases to 0, χ_θ decreases to $\chi_0=\mathrm{id}$ and φ_θ decreases to φ. The following lemma gives some properties of φ_θ where the coordinates (x_1', y, v') are introduced in Section 2.1.

Lemma 2.4.1. *There is a constant $c>0$ such that for $\theta\in\mathbb{C}^*$ small enough, $dd^c\varphi_\theta+\gamma$ vanishes on $\{|x_1|>c|\theta|\}$. Moreover, we can write*

$$dd^c\varphi_\theta+\gamma = A dx_1\wedge d\overline{x}_1 + B$$

where A is a smooth function such that $\|A\|_\infty\leq c|\theta|^{-2}$ and B is a smooth form such that $\|B\|_\infty\leq c|\theta|^{-1}$.

Proof. Since $\widehat{\Delta}$ is given by $x_1=0$ and $dd^c\varphi=[\widehat{\Delta}]-\gamma$, the function $\psi:=\varphi-\log|x_1|$ is smooth. By definition, $\varphi_\theta=\varphi$ on $\{\varphi>\log|\theta|\}$ which contains $\{|x_1|>c|\theta|\}$ for some constant $c>0$ large enough. So, we have for $|x_1|>c|\theta|$

$$dd^c\varphi_\theta+\gamma = dd^c\varphi+\gamma = 0.$$

This proves the first assertion of the lemma.

For the second assertion, observe that φ_θ is constant on $\{|x_1| < c'|\theta|\}$ for some constant $c' > 0$. Therefore, it is enough to consider the problem on the domain $\{c'|\theta| \leq |x_1| \leq c|\theta|\}$ where we have $dd^c\varphi = \gamma$. Observe that $\|\varphi\|_{\mathscr{C}^1} \lesssim |x_1|^{-1}$, $\|\varphi\|_{\mathscr{C}^2} \lesssim |x_1|^{-2}$ and that the derivatives of χ_θ are bounded by a constant independent of θ. We have

$$dd^c\varphi_\theta = dd^c\chi_\theta(\varphi) = \chi''_\theta(\varphi)d\varphi \wedge d^c\varphi + \chi'_\theta(\varphi)dd^c\varphi.$$

The last term is bounded. For the first term, we have since ψ is smooth

$$d\varphi \wedge d^c\varphi = d(\log|x_1| + \psi) \wedge d^c(\log|x_1| + \psi) = \frac{i}{\pi}|x_1|^{-2}dx_1 \wedge d\overline{x}_1 + O(|x_1|^{-1}).$$

This implies the result. □

Lemma 2.4.2. *The function* $(\theta, z) \mapsto \varphi_\theta(z)$ *can be extended to a quasi-p.s.h. function on* $\mathbb{C} \times \widehat{X \times X}$ *which is continuous outside* $\{0\} \times \widehat{\Delta}$ *and d.s.h. on* $\mathbb{P}^1 \times \widehat{X \times X}$*. We have* $\varphi_0(z) = \varphi(z)$ *and* $dd^c\varphi_\theta(z) \geq -\lambda\widehat{\omega}(z)$ *on* $\mathbb{C} \times \widehat{X \times X}$ *for some constant* $\lambda > 0$*. Moreover,* $dd^c\varphi_\theta(z)$ *can be written as a difference of positive closed currents on* $\mathbb{P}^1 \times \widehat{X \times X}$ *which are smooth on* $\mathbb{C}^* \times \widehat{X \times X}$*.*

Proof. If $\psi(\theta, z) := \varphi(z) - \log|\theta|$, then we have on $\mathbb{C}^* \times \widehat{X \times X}$

$$\begin{aligned} dd^c\varphi_\theta(z) &= [\chi'(\psi)]^2 d\psi \wedge d^c\psi + \chi''(\psi)dd^c\psi \\ &\geq \chi''(\psi)dd^c\psi = \chi''(\psi)dd^c\varphi(z). \end{aligned}$$

Hence, $dd^c\varphi_\theta(z) \geq -\lambda\widehat{\omega}(z)$, $\lambda > 0$, on $\mathbb{C}^* \times \widehat{X \times X}$ because χ'' is positive bounded and φ is quasi-p.s.h. On the other hand, by definition, $\varphi_\theta(z) = \log|\theta| - 1$ when $|\theta| \geq 1$ and $\varphi_\theta(z)$ is bounded from above when $|\theta| \leq 1$. By classical properties of p.s.h. functions, $\varphi_\theta(z)$ can be extended to a quasi-p.s.h. function and the estimate $dd^c\varphi_\theta(z) \geq -\lambda\widehat{\omega}(z)$ holds on $\mathbb{C} \times \widehat{X \times X}$.

Since $\varphi_\theta(z) = \log|\theta| - 1$ for $|\theta| \geq 1$, $\varphi_\theta(z)$ is d.s.h. on $\mathbb{P}^1 \times \widehat{X \times X}$. We have for the dd^c operator on $\mathbb{P}^1 \times \widehat{X \times X}$

$$dd^c\varphi_\theta(z) \geq -\big[\{\infty\} \times \widehat{X \times X}\big] - \lambda\widehat{\omega}(z).$$

So, we can write $dd^c\varphi_\theta(z)$ as the following difference of two positive closed currents

$$\Big(dd^c\varphi_\theta(z) + \big[\{\infty\} \times \widehat{X \times X}\big] + \lambda\widehat{\omega}(z)\Big) - \Big(\big[\{\infty\} \times \widehat{X \times X}\big] + \lambda\widehat{\omega}(z)\Big).$$

These currents are smooth on $\mathbb{C}^* \times \widehat{X \times X}$ since $\varphi_\theta(z)$ is smooth there.

It remains to study $\varphi_\theta(z)$ when $\theta = 0$ or $\theta \to 0$. For $a \notin \widehat{\Delta}$, we have $\varphi_\theta(z) \to \varphi(a)$ when $(\theta, z) \to (0, a)$. Therefore, $\varphi_\theta(z)$ is continuous out of $\{0\} \times \widehat{\Delta}$ and $\varphi_0(z) = \varphi(z)$ outside $\widehat{\Delta}$. Finally, since $\varphi_\theta \leq \max(\varphi, \log|\theta|)$, we have that $\varphi_\theta(z) \to -\infty$ when (θ, z) tends to $\{0\} \times \widehat{\Delta}$. Since $\varphi_\theta(z)$ is quasi-p.s.h. on $\mathbb{C} \times \widehat{X \times X}$, we deduce that $\varphi_0(z) = -\infty = \varphi(z)$ on $\widehat{\Delta}$. □

Define for $\theta \in \mathbb{C}^*$ the current Θ_θ on $\widehat{X \times X}$ by

$$\Theta_\theta := (dd^c \varphi_\theta + \gamma) \wedge \eta \quad \text{and} \quad \Theta_0 := [\widehat{\Delta}] \wedge \eta,$$

see Example 2.3.1. Observe that $\Theta_\theta = \gamma \wedge \eta$ for $|\theta| > 1$ since in this case φ_θ is constant. So, define also $\Theta_\infty := \gamma \wedge \eta$. Since γ and η are smooth, they can be written as differences of smooth positive closed forms. By Lemma 2.4.2, for each $\theta \neq 0$, $dd^c \varphi_\theta$ can be written as a difference of smooth positive closed forms on X with masses bounded by a constant independent of θ. Therefore, we can write $\Theta_\theta := \Theta_\theta^+ - \Theta_\theta^-$ where $\Theta_\theta^\pm$ are smooth positive closed with mass bounded by a constant independent of θ. We see that the family $(\Theta_\theta)_{\theta \in \mathbb{P}^1}$ is continuous with respect to the $*$-topology. Define

$$\mathcal{L}_\theta^\pm := \mathcal{L}_{\Theta_\theta^\pm} \quad \text{and} \quad \mathcal{L}_\theta := \mathcal{L}_{\Theta_\theta} = \mathcal{L}_\theta^+ - \mathcal{L}_\theta^-.$$

Note that $\mathcal{L}_\theta$ is symmetric.

• **Regularization of currents.** Regularization of positive closed currents on complex manifolds was developed by Demailly in the case of bidegree $(1,1)$ [10]. The case of bidegree (p,p) was studied in [19]. Define $S_\theta := \mathcal{L}_\theta(S)$ for all currents S in $\mathcal{D}_p$.

Lemma 2.4.3. *The current S_θ depends continuously on (θ, S) for the $*$-topology on S, S_θ. In particular, we have $\|S_\theta\|_* \leq c\|S\|_*$ for some constant $c > 0$ independent of S and θ. Moreover, we have $\operatorname{dist}_2(S_\theta, S) \leq c|\theta|\|S\|_*$ with $c > 0$ independent of S and θ.*

Proof. The estimate $\|S_\theta\|_* \leq c\|S\|_*$ is clear for $\theta = 0$ since $\mathcal{L}_0 = \mathrm{id}$, see Example 2.3.1. The case $\theta \neq 0$ is a consequence of Lemma 2.3.4 applied to $\mathcal{L}_\theta^\pm$. When θ tends to $a \in \mathbb{C}^*$, then φ_θ converges in the $\mathcal{C}^\infty$-topology to φ_a. Therefore, S_θ depends continuously on (θ, S) for $\theta \in \mathbb{C}^*$.

It remains to prove the estimate on $\operatorname{dist}_2(S_\theta, S)$ for $|\theta| \leq 1$. This and the triangle inequality imply the continuity of S_θ at $\theta = 0$, see also Proposition 2.2.1. We can assume that S is positive and that $\|S\| \leq 1$. Let Φ be a test form such that $\|\Phi\|_{\mathcal{C}^2} \leq 1$. We have using the description of $\mathcal{L}_0$ in Example 2.3.1

$$\begin{aligned}
\langle S_\theta - S, \Phi \rangle &= \langle dd^c(\varphi_\theta - \varphi) \wedge \eta \wedge \Pi_1^*(S), \Pi_2^*(\Phi) \rangle \\
&= \langle (\varphi_\theta - \varphi)\eta \wedge \Pi_1^*(S), \Pi_2^*(dd^c\Phi) \rangle \\
&= \big\langle S, (\Pi_1)_*\big((\varphi_\theta - \varphi)\eta \wedge \Pi_2^*(dd^c\Phi)\big) \big\rangle.
\end{aligned}$$

We have to bound the last integral by $c|\theta|$ for some constant $c > 0$.

Since $\|S\| \leq 1$, it is enough to show that the form $(\Pi_1)_*\big[(\varphi_\theta - \varphi)\eta \wedge \Pi_2^*(dd^c\Phi)\big]$ has a $\|\cdot\|_\infty$-norm bounded by $c|\theta|$. The map Π_1 is a submersion. So, the coefficients of the considered form are equal to some integrals of coefficients of $(\varphi_\theta - \varphi)\eta \wedge \Pi_2^*(dd^c\Phi)$ on fibers of Π_1. The $\|\cdot\|_\infty$ estimate is not difficult to obtain. Indeed, $\eta \wedge \Pi_2^*(dd^c\Phi)$ is a smooth form with bounded $\|\cdot\|_\infty$-norm, the function $\varphi_\theta - \varphi$ has support in a neighbourhood of $\widehat{\Delta}$ of size $\lesssim |\theta|$ and satisfies $|\varphi_\theta - \varphi| \lesssim -\log \operatorname{dist}(\cdot, \widehat{\Delta})$ near $\widehat{\Delta}$. □

We deduce the following result obtained in [19], see also Propositions 3.1.2 and 3.2.8 below.

Theorem 2.4.4. *Smooth forms are dense in $\mathcal{D}_p$ and in $\mathcal{D}_p^0$ for the $*$-topology. Moreover, there is a constant $c > 0$ such that for every current $S \in \mathcal{D}_p$, we can write $S = S^+ - S^-$ with $S^\pm \in \mathcal{C}_p$, $\|S^\pm\| \leq c\|S\|_*$ and $S^\pm$ approximable by smooth forms in $\mathcal{C}_p$.*

Proof. We prove the first assertion. If S is in $\mathcal{D}_p$, we can add to S a smooth form in order to obtain a current in $\mathcal{D}_p^0$. So, it is enough to approximate currents S in $\mathcal{D}_p^0$ by smooth forms in $\mathcal{D}_p^0$. Observe that the problem is easy when S is a form of class $\mathcal{C}^1$. Indeed, we can write $S = dd^cU$ with U of class $\mathcal{C}^2$ and approximate S uniformly by $S_\epsilon := dd^cU_\epsilon$ where U_ϵ is smooth and $U_\epsilon \to U$ in the $\mathcal{C}^2$ topology. It remains to approximate S by $\mathcal{C}^1$ forms in $\mathcal{D}_p^0$. Consider non-zero complex numbers $\theta_1, \ldots, \theta_{k+2}$. The currents Θ_{θ_i} are smooth, then the associated transforms $\mathcal{L}_{\theta_i}$ are regular. By Lemma 2.4.3, we can choose θ_i converging to 0 such that $\mathcal{L}_{\theta_{k+2}} \circ \cdots \circ \mathcal{L}_{\theta_1}(S)$ converges to S. By Corollary 2.3.3 and Lemma 2.3.4, $\mathcal{L}_{\theta_{k+2}} \circ \cdots \circ \mathcal{L}_{\theta_1}(S)$ is a $\mathcal{C}^1$ form in $\mathcal{D}_p^0$. This completes the proof of the first assertion.

For the second assertion, we can assume that S is positive. Recall that when $\theta \neq 0$, $\mathcal{L}_\theta = \mathcal{L}_\theta^+ - \mathcal{L}_\theta^-$ where $\mathcal{L}_\theta^\pm$ are associated to smooth positive forms $\Theta_\theta^\pm$ with mass bounded by a constant. Therefore, $\mathcal{L}_{\theta_{k+2}} \circ \cdots \circ \mathcal{L}_{\theta_1}(S)$ is a difference of $\mathcal{C}^1$ positive closed forms of bounded mass. We obtain the result by extracting subsequences of forms converging to some currents $S^\pm$. □

Corollary 2.4.5. *Let S be a current in $\mathcal{D}_p$ and S' a current in $\mathcal{D}_{p'}$ with $p+p' \leq k$. Assume that S restricted to an open set W is a continuous form. Then $S \wedge S'$ is defined on W and its mass on W satisfies*

$$\|S \wedge S'\|_W \leq c\|S\|_*\|S'\|_*$$

for some constant $c > 0$ independent of S and S'.

Proof. It is clear that $S \wedge S'$ is well-defined on W and depends continuously on S' for the $*$-topology on S'. Therefore, by Theorem 2.4.4, we can assume that S' is positive and smooth. Now $S \wedge S'$ is defined on X for every S smooth or not. We can assume that S is positive but we may loose the continuity of S. The current $S \wedge S'$ is positive on X. Its mass can be computed cohomologically. Therefore, we have

$$\|S \wedge S'\|_W \leq \|S \wedge S'\| \leq c\|S\|\|S'\|.$$

This implies the result. □

Lemma 2.4.6. *Let S and S_θ be as above. Assume that S is smooth. Then S_θ is smooth for every θ and*

$$\|S_\theta - S\|_\infty \leq c|\theta|\|S\|_{\mathcal{C}^1}$$

where $c > 0$ is a constant independent of S and θ.

Proof. The current S_0 is equal to S. Hence, S_0 is smooth. For $\theta \neq 0$, $\mathcal{L}_\theta$ is a regular transform. Using the fact that Π_2 is a submersion, we deduce easily that $S_\theta = \mathcal{L}_\theta(S)$ is smooth. It remains to prove the estimate in the lemma.

Assume that $\|S\|_{\mathcal{C}^1} \leq 1$. Observe that $(\Pi_2)_*(\Theta_\theta)$ is a closed current of bidegree $(0,0)$ on X. So, it is defined by a constant function. On the other hand, since Θ_θ is cohomologous to Θ_0, $(\Pi_2)_*(\Theta_\theta)$ is cohomologous to

$$(\Pi_2)_*(\Theta_0) = (\pi_2)_*\pi_*(\Theta_0) = (\pi_2)_*[\Delta] = [X].$$

Hence, $(\Pi_2)_*(\Theta_\theta)$ is equal to 1. We deduce that $S = (\Pi_2)_*(\Pi_2^*(S) \wedge \Theta_\theta)$ and then

$$S_\theta - S = (\Pi_2)_*(\pi^*(S') \wedge \Theta_\theta), \quad \text{where} \quad S' := \pi_1^*(S) - \pi_2^*(S).$$

Observe that $\|S'\|_{\mathcal{C}^1}$ is bounded and the restriction of S' to Δ vanishes. If $S'' := \pi^*(S')$, then $\|S''\|_{\mathcal{C}^1}$ is bounded and S'' restricted to $\widehat{\Delta}$ vanishes. Therefore, in the local coordinates near $\widehat{\Delta}$ as in Section 2.1, we have

$$S''(x_1, y, v') = |x_1|A + Bdx_1 + Cd\overline{x}_1$$

where A, B, C are bounded forms.

The coefficients of $S_\theta - S$ at a point y^0 is computed by some integrals involving the coefficients of $S''\wedge\Theta_\theta = S''\wedge(dd^c\varphi_\theta+\gamma)\wedge\eta$ on $\{y = y^0\}$. The above description of S'' together with Lemma 2.4.1, implies that these coefficients are $\lesssim |\theta|$. The result follows. □

• **Green potential and dd^c-equation.** Consider a current S in $\mathcal{D}_p^0$ with $p \geq 1$. Then, since $[S] = 0$, by dd^c-lemma [10, 46], there is a real current U_S of bidegree $(p-1, p-1)$ such that $dd^c U_S = S$. We call U_S *a potential* of S. In order to construct an explicit potential and to estimate its norm, we use a transform of bidegree $(-1,-1)$. Choose a real smooth $(k-1,k-1)$-form β on $\widehat{X \times X}$ such that $dd^c\beta = -\gamma\wedge\eta + \pi^*(\alpha_\Delta)$ where α_Δ, γ and η are introduced in Section 2.1. We can choose β symmetric. Consider the symmetric transform $\mathcal{L}_K$ with $K := \varphi\eta - \beta$. The following result was obtained in [20, Proposition 2.1], see also [32].

Proposition 2.4.7. *Let S be a current in $\mathcal{D}_p^0$ with $p \geq 1$. Then $U_S := \mathcal{L}_K(S)$ is a potential of S. Moreover, we have*

$$\|U_S\|_{L^{1+1/k}} \leq c\|S\|_*$$

for some constant $c > 0$ independent of S.

Proof. By Proposition 2.3.2, $S \mapsto \mathcal{L}_K(S)$ is continuous with respect to the $*$-topology on $S \in \mathcal{D}_p^0$ and the estimate on $\|U_S\|_{L^{1+1/k}}$ is clear. We show that $dd^c U_S = S$. By Theorem 2.4.4, it is enough to consider S smooth. Define $K' := \pi_*(K)$. This is a form smooth outside Δ. We have seen in Section 2.3 that

$$|K'| \lesssim -\log \operatorname{dist}(\cdot, \Delta) \operatorname{dist}(\cdot, \Delta)^{2-2k}$$

near Δ. We also have

$$dd^c K' = \pi_*(dd^c K) = \pi_*\big([\widehat{\Delta}] \wedge \eta - \pi^*(\alpha_\Delta)\big) = [\Delta] - \alpha_\Delta.$$

So, K' is a kernel for solving the dd^c-equation on X. Since S is smooth, we have $U_S = (\pi_2)_*(\pi_1^*(S) \wedge K')$ and

$$\begin{aligned} dd^c U_S &= (\pi_2)_*\big(\pi_1^*(S) \wedge [\Delta]\big) - (\pi_2)_*\big(\pi_1^*(S) \wedge \alpha_\Delta\big) \\ &= S - (\pi_2)_*\big(\pi_1^*(S) \wedge \alpha_\Delta\big) = S, \end{aligned}$$

where the last identity is obtained using that $[S] = 0$ and that α_Δ is a combination of forms of type $\beta(x) \wedge \beta'(y)$ with β and β' closed. □

Definition 2.4.8. We call $\mathcal{L}_K(S)$ *the Green potential of* S.

Note that the Green potential depends on the choice of K.

Remark 2.4.9. The transform associated to $\frac{i}{\pi}\partial K$ solves the $\overline{\partial}$-equation on X and $\frac{i}{\pi}\pi_*(\partial K)$ is a kernel which solves the $\overline{\partial}$-equation.

3 Structural varieties and super-potentials

In this section, we define for each positive closed (p,p)-current a super-potential which is a function on the space of smooth forms in $\mathcal{D}_{k-p+1}^0$. In this space, we construct some special structural lines parametrized by the projective line $\mathbb{P}^1$. The restriction of the super-potential to such a structural line is a d.s.h. function. This is a key point in our study. We will also consider currents with regular super-potentials and their intersection.

3.1 Structural varieties in the space of currents

Consider a current $\mathcal{R}$ on $V \times X$ which is a difference of two positive closed (s,s)-currents. We will use in next sections the case where $s = k-p+1$. By Proposition 2.2.5, for θ in V outside a locally pluripolar set, the slice $R_\theta := \langle \mathcal{R}, \pi_V, \theta\rangle$ is well-defined and is a current in $\mathcal{D}_s$. Its cohomology class does not depend on θ. Assume that R_θ is in $\mathcal{D}_s^0$, i.e. $[R_\theta] = 0$. So, we obtain a map $\tau : V \to \mathcal{D}_s^0$ given by $\theta \mapsto R_\theta$ which is defined out of a locally pluripolar set.

Definition 3.1.1. We say that τ or the family $(R_\theta)_{\theta\in V}$ defines a *structural variety* of $\mathcal{D}^0_s$. When R_θ is defined for every θ and depends continuously on θ for the $*$-topology, we say that the structural variety is *continuous*.

In what follows, we use some *structural lines*, i.e. structural varieties parametrized by the projective line $\mathbb{P}^1 = \mathbb{C}\cup\{\infty\}$. Let $\mathcal{L}_\theta := \mathcal{L}_{\Theta_\theta}$ be transforms defined in Section 2.4. For a given current R in $\mathcal{D}^0_s$ and for $\theta \in \mathbb{C}\cup\{\infty\}$, consider the current $R_\theta := \mathcal{L}_\theta(R)$. Recall that $\mathcal{L}_\theta$, Θ_θ, R_θ do not depend on θ when $|\theta| \geq 1$ and that $\mathcal{L}_0 = \mathrm{id}$, $R_0 = R$.

Proposition 3.1.2. *The family of currents $(R_\theta)_{\theta\in\mathbb{P}^1}$ defines a continuous structural line in $\mathcal{D}^0_s$ which depends linearly on R. Moreover, there is a constant $c > 0$ independent of R such that $\|R_\theta\|_* \leq c\|R\|_*$ for every θ.*

Definition 3.1.3. We call $(R_\theta)_{\theta\in\mathbb{P}^1}$ *the special structural line* associated to R.

Proof of Proposition 3.1.2. The linear dependence on R is clear. The continuity of $(R_\theta)_{\theta\in\mathbb{P}^1}$ and the estimate on $\|R_\theta\|_*$ are proved in Lemma 2.4.3. Let τ_0 denote the projection of $\mathbb{P}^1 \times \widehat{X\times X}$ on $\mathbb{P}^1$ and τ the projection on $\widehat{X\times X}$. Consider $\varphi_\theta(z)$ as a function on $\mathbb{C}\times\widehat{X\times X}$. Define a current $\widehat{\mathcal{R}}$ on $\mathbb{C}^*\times\widehat{X\times X}$ by

$$\widehat{\mathcal{R}}(\theta, z) := \big(dd^c\varphi_\theta(z) + \tau^*(\gamma)\big)\wedge\tau^*(\eta)\wedge\tau^*(\Pi_1^*(R)).$$

In this wedge-product, each current is a difference of positive closed currents with bounded mass in $\mathbb{P}^1\times\widehat{X\times X}$. We can apply Corollary 2.4.5 to the current $\widehat{\mathcal{R}}$, which is well-defined on $\mathbb{C}^*\times\widehat{X\times X}$, and Skoda's theorem [44] on the extension of positive closed currents. Hence, the trivial extension of $\widehat{\mathcal{R}}$ is a difference of positive closed currents on $\mathbb{P}^1\times\widehat{X\times X}$ with bounded mass. Denote also by $\widehat{\mathcal{R}}$ this extension.

On $\mathbb{C}^*\times\widehat{X\times X}$, in the definition of $\widehat{\mathcal{R}}$, all currents except R, are smooth. We deduce easily from the slicing theory that

$$\langle\widehat{\mathcal{R}}, \tau_0, \theta\rangle = (dd^c_z\varphi_\theta + \gamma)\wedge\eta\wedge\Pi_1^*(R)$$

where we identify $\{\theta\}\times\widehat{X\times X}$ with $\widehat{X\times X}$. Let $\tau_2 := (\tau_0, \Pi_2\circ\tau)$ denote the projection of $\mathbb{P}^1\times\widehat{X\times X}$ onto the product of $\mathbb{P}^1$ with the second factor X. Define $\mathcal{R} := (\tau_2)_*(\widehat{\mathcal{R}})$. It is deduced from the slicing theory that

$$\langle\mathcal{R}, \pi_{\mathbb{P}^1}, \theta\rangle = (\Pi_2)_*\langle\widehat{\mathcal{R}}, \tau_0, \theta\rangle = R_\theta,$$

for $\theta\in\mathbb{C}^*$. Recall that R_θ depends continuously on $\theta\in\mathbb{P}^1$. By Remark 2.2.6, the identity $\langle\mathcal{R}, \pi_{\mathbb{P}^1}, \theta\rangle = R_\theta$ holds for any $\theta\in\mathbb{P}^1$. So, $(R_\theta)_{\theta\in\mathbb{P}^1}$ is a structural line. □

Remark 3.1.4. We can prove that $\theta\mapsto\mathcal{L}_\theta^{k+2}(R)$ defines a continuous structural line. In this case, for $\theta\neq 0$, $\mathcal{L}_\theta^{k+2}(R)$ is a $\mathcal{C}^1$ form.

3.2 Super-potentials of currents

Consider a current S in $\mathcal{D}_p$. The super-potentials of S are defined (at least) on the smooth forms in $\mathcal{D}^0_{k-p+1}$. They are unique under apropriate normalization.

Let $\alpha = \{\alpha_1, \ldots, \alpha_h\}$ with $h := \dim H^{p,p}(X,\mathbb{R})$ be a fixed family of real smooth closed (p,p)-forms such that the family of classes $[\alpha] = \{[\alpha_1], \ldots, [\alpha_h]\}$ is a basis of $H^{p,p}(X,\mathbb{R})$. We can find a family $\alpha^\vee = \{\alpha_1^\vee, \ldots, \alpha_h^\vee\}$ of real smooth closed $(k-p,k-p)$-forms such that $[\alpha^\vee] = \{[\alpha_1^\vee], \ldots, [\alpha_h^\vee]\}$ is the dual basis of $[\alpha]$ with respect to the cup-product $\smile$. Let R be a current in $\mathcal{D}^0_{k-p+1}$ and U_R a potential of R. Adding to U_R a suitable combination of $\alpha_i^\vee$ allows to assume that $\langle U_R, \alpha_i\rangle = 0$ for $i = 1, \ldots, h$. We say that U_R is *α-normalized*.

Lemma 3.2.1. *Assume that S is smooth or that R, U_R are smooth. Then $\langle S, U_R\rangle$ does not depend on the choice of U_R. Assume that $[S] = 0$. Let U_S be a potential of S, smooth if S is smooth. Let U_R' be another potential of R, smooth when R is smooth. Then $\langle S, U_R'\rangle = \langle S, U_R\rangle = \langle U_S, R\rangle$. In particular, $\langle S, U_R\rangle$ does not depend on α and $\alpha^\vee$.*

Proof. Let U_R' be another α-normalized potential of R. We have $dd^c(U_R - U_R') = 0$ and $[\alpha_i] \smile [U_R - U_R'] = 0$ for every i. Since $[\alpha]$ is a basis of $H^{p,p}(X,\mathbb{R})$, we deduce that $[S] \smile [U_R - U_R'] = 0$. Hence, $\langle S, U_R\rangle = \langle S, U_R'\rangle$. So, $\langle S, U_R\rangle$ does not depend on the choice of U_R. If $[S] = 0$, we have

$$\langle S, U_R'\rangle = \langle dd^c U_S, U_R'\rangle = \langle U_S, dd^c U_R'\rangle = \langle U_S, R\rangle.$$

These identities hold for all U_R' not necessarily normalized, in particular for U_R. Note that the smoothness of S, U_S or R, U_R, U_R' implies that the considered integrals are meaningful. □

Definition 3.2.2. *The α-normalized super-potential $\mathcal{U}_S$ of S*, is the following function defined on smooth forms R in $\mathcal{D}^0_{k-p+1}$ by

$$\mathcal{U}_S(R) := \langle S, U_R\rangle, \tag{3.1}$$

where U_R is an α-normalized smooth potential of R. We say that S has *a continuous super-potential*[4] if $\mathcal{U}_S$ can be extended to a function on $\mathcal{D}^0_{k-p+1}$ which is continuous with respect to the $*$-topology. In this case, the extension is also denoted by $\mathcal{U}_S$ and is also called super-potential of S.

Note that the α-normalized super-potential of α_i is identically zero. By Lemma 3.2.1, when $[S] = 0$, the super-potential $\mathcal{U}_S$ does not depend on the choice of α. When S is smooth then S has a continuous super-potential and the formula (3.1) holds for all R in $\mathcal{D}^0_{k-p+1}$. In this case, if $[S] = 0$ and if U_S is a smooth potential of S, we also have $\mathcal{U}_S(R) = \langle U_S, R\rangle$.

[4] this is equivalent to the notion of PC current introduced in [20].

Proposition 3.2.3. *Let S and S' be two currents in $\mathcal{D}_p$ such that $[S] = [S']$. If they have the same α-normalized super-potential then they are equal.*

Proof. The α-normalized super-potential $\mathcal{U}_{S''}$ of $S'' := S - S'$ vanishes identically. If U is a real smooth $(k-p, k-p)$-form, then U is a potential of $dd^c U$ which is a form in $\mathcal{D}^0_{k-p+1}$. Since $[S''] = 0$, it follows from Lemma 3.2.1, that $\langle S'', U\rangle = \mathcal{U}_{S''}(dd^c U) = 0$. Hence, $S'' = 0$. □

Here is one of the fundamental properties of super-potential. It can be extended to more general structural varieties but, for simplicity we restrict ourselves to this particular case.

Proposition 3.2.4. *Let $(R_\theta)_{\theta\in\mathbb{P}^1}$ be the special structural line associated to a smooth form $R \in \mathcal{D}^0_{k-p+1}$. Let S be a current in $\mathcal{D}_p$. Then $\theta \mapsto \mathcal{U}_S(R_\theta)$ is a continuous d.s.h. function on $\mathbb{P}^1$ which is constant on $\{|\theta| \geq 1\}$. Moreover, we have*

$$\|dd^c_\theta \mathcal{U}_S(R_\theta)\|_* \leq c\|S\|_*\|R\|_*$$

where $c > 0$ is a constant independent of R and S.

Proof. By Lemma 2.4.6 applied to R_θ, the function $H(\theta) := \mathcal{U}_S(R_\theta)$ is continuous on $\mathbb{P}^1$. It remains to bound the mass of $dd^c H$. Since this function depends continuously on S, by Theorem 2.4.4, we can assume that S is smooth. Recall that the α-normalized super-potential of α_i is zero. Subtracting from S a combination of α_i allows to assume that $[S] = 0$. So, we can use the last assertion of Lemma 3.2.1: if U is a smooth potential of S, then $H(\theta) = \langle U, R_\theta\rangle$.

It is enough to estimate the mass of $dd^c H$ on $\mathbb{C}^*$. Indeed, the continuity of H implies that $dd^c H$ has no mass on finite sets. Consider in $\mathbb{P}^1 \times \widehat{X \times X}$ the currents

$$\widehat{\mathcal{R}}_U := \widehat{\mathcal{R}} \wedge \tau^*\Pi_2^*(U) \quad \text{and} \quad \widehat{\mathcal{R}}_S := \widehat{\mathcal{R}} \wedge \tau^*\Pi_2^*(S).$$

These currents are smooth on $\mathbb{C}^* \times \widehat{X \times X}$. A direct computation gives $H = (\tau_0)_*(\widehat{\mathcal{R}}_U)$ and $dd^c H = (\tau_0)_*(\widehat{\mathcal{R}}_S)$ on $\mathbb{C}^*$. So, it is enough to estimate the mass of $\widehat{\mathcal{R}}_S$ on $\mathbb{C}^* \times \widehat{X \times X}$. By Corollary 2.4.5, since S and R are smooth, this mass is bounded by a constant times

$$\|\widehat{\mathcal{R}}\|_*\|\tau^*\Pi_2^*(S)\|_* \lesssim \|\tau^*\Pi_1^*(R)\|_*\|\tau^*\Pi_2^*(S)\|_* \lesssim \|R\|_*\|S\|_*,$$

where the last inequality follows from the fact that τ, Π_1, Π_2 are submersions. □

Lemma 3.2.5. *Let $\mathcal{U}_{S_\theta}$ be the α-normalized super-potential of S_θ. If $[S] = 0$, then $\mathcal{U}_{S_\theta}(R) = \mathcal{U}_S(R_\theta)$ for R smooth.*

Proof. Since $S_\theta = \mathcal{L}_\theta(S)$, the Green potential of S_θ is equal to $\mathcal{L}_K\mathcal{L}_\theta(S)$. Using the symmetry of $\mathcal{L}_\theta$ and $\mathcal{L}_K$, we have by Lemma 3.2.1

$$\mathcal{U}_{S_\theta}(R) = \langle \mathcal{L}_K\mathcal{L}_\theta(S), R\rangle = \langle S, \mathcal{L}_\theta\mathcal{L}_K(R)\rangle.$$

On the other hand,

$$dd^c\mathcal{L}_\theta\mathcal{L}_K(R) = \mathcal{L}_\theta(dd^c\mathcal{L}_K(R)) = \mathcal{L}_\theta(R) = R_\theta.$$

It follows that $\mathcal{U}_{S_\theta}(R) = \mathcal{U}_S(R_\theta)$. □

The following result is an analogue of the estimate in Proposition 2.2.3 for super-potentials of currents.

Theorem 3.2.6. *Let S be a current in $\mathcal{D}_p$ and $\mathcal{U}_S$ the α-normalized super-potential of S. Then we have for R smooth in $\mathcal{D}^0_{k-p+1}$ with $\|R\|_* \leq 1$*

$$|\mathcal{U}_S(R)| \leq c\|S\|_*\big(1 + \log^+ \|R\|_{\mathcal{C}^1}\big),$$

where $\log^+ := \max(\log, 0)$ and $c > 0$ is a constant independent of S, R.

Proof. Subtracting from S a combination of α_i allows to assume that $[S] = 0$. We can also assume that $\|S\|_* = 1$. Let U_S be the Green potential of S. By Lemma 3.2.1, $\mathcal{U}_S(R) = \langle U_S, R\rangle$. We have to show that

$$M_{S,R} := \frac{|\langle U_S, R\rangle|}{1 + \log^+ \|R\|_{\mathcal{C}^1}}$$

is bounded when $\|R\|_* \leq 1$. The proof uses Proposition 2.3.2, Lemma 2.2.4 and special structural lines in $\mathcal{D}^0_{k-p+1}$. Consider the numbers $q_n \geq 1$ given by the induction identity $q_n^{-1} = q_{n-1}^{-1} - 1 + (1 + 1/k)^{-1}$ for $n \leq k + 1$ with $q_0 = 1$. We have $q_{k+1} = \infty$.

Claim. *For every $0 \leq n \leq k + 1$ and $M > 0$, there is a constant $c > 0$ independent of S, R such that $M_{S,R} \leq c$ if $\|R\|_* \leq 1$ and $\|R\|_{L^{q_n}} \leq M$.*

For $n = 0$, the claim implies the theorem, i.e. $M_{S,R}$ is bounded when $\|R\|_* \leq 1$. Indeed, we have $\|R\|_{L^1} \lesssim \|R\| \lesssim \|R\|_*$ and then the hypothesis $\|R\|_{L^{q_0}} \leq M$ is satisfied. We prove now the claim using a decreasing induction on n. For $n = k + 1$, by Proposition 2.3.2, $\|U_S\|_{L^1}$ is bounded uniformly on S. If $\|R\|_\infty$ is bounded, it is clear that $\langle U_S, R\rangle$ is bounded. So, the claim is true for $n = k + 1$. Assume now that the claim is true for $n + 1$. We check it for n and we only have to consider the case where $\|R\|_{\mathcal{C}^1}$ is large.

Let R_θ be as above and define $H_{S,R}(\theta) := \mathcal{U}_S(R_\theta)$. By Proposition 3.2.4, $H_{S,R}$ is a continuous d.s.h. function on $\mathbb{P}^1$. It is equal to some constant $c_{S,R}$ on $\{|\theta| \geq 1\}$. We have $c_{S,R} = \langle U_S, R_\infty\rangle$. Moreover, $\|dd^c H_{S,R}\|$ is bounded uniformly on S, R. On the other hand, by Propositions 2.3.2 and 3.1.2, R_∞ is a smooth form in $\mathcal{D}^0_{k-p+1}$ with bounded $L^{q_{n+1}}$-norm and bounded $\|\cdot\|_*$-norms. Since $\langle U_S, R\rangle$ depends linearly on R, we can apply the claim to R_∞ and deduce that

$$c_{S,R} \lesssim 1 + \log^+ \|R_\infty\|_{\mathcal{C}^1} \lesssim 1 + \log^+ \|R\|_{\mathcal{C}^1}.$$

Because $\langle U_S, R\rangle = H_{S,R}(0)$, it is enough to show that

$$|H_{S,R}(0) - c_{S,R}| \lesssim 1 + \log^+ \|R\|_{\mathcal{C}^1}.$$

Since $\|dd^c H_{S,R}\|$ is uniformly bounded, by Lemma 2.2.4 applied to $H_{S,R} - c_{S,R}$, there is a θ with $|\theta| \leq \|R\|_{\mathcal{C}^1}^{-1}$ such that

$$|H_{S,R}(\theta) - c_{S,R}| \lesssim 1 + \log^+ \|R\|_{\mathcal{C}^1}.$$

On the other hand, Lemma 2.4.6 implies that

$$\|R - R_\theta\|_\infty \lesssim |\theta|\|R\|_{\mathcal{C}^1} \leq 1.$$

Therefore, using that $\|U_S\|_{L^1}$ is bounded, we obtain

$$\begin{aligned} |H_{S,R}(0) - c_{S,R}| &\leq |H_{S,R}(0) - H_{S,R}(\theta)| + |H_{S,R}(\theta) - c_{S,R}| \\ &= |\langle U_S, R - R_\theta\rangle| + |H_{S,R}(\theta) - c_{S,R}| \\ &\lesssim 1 + \log^+ \|R\|_{\mathcal{C}^1}. \end{aligned}$$

This completes the proof. □

We will use the following notion of convergence.

Definition 3.2.7. Let (S_n) be a sequence of currents converging in $\mathcal{D}_p$ to a current S. Let $\mathcal{U}_S$, $\mathcal{U}_{S_n}$ be the α-normalized super-potentials of S, S_n. We say that the convergence is *SP-uniform* if $\mathcal{U}_{S_n}$ converge to $\mathcal{U}_S$ uniformly on any $*$-bounded set of smooth forms in $\mathcal{D}^0_{k-p+1}$.

By linearity, it is enough to check the SP-uniform convergence on the unit ball of $\mathcal{D}^0_{k-p+1}$. This notion does not depend on α. Indeed, by Lemma 3.2.1, the case where $[S_n] = [S] = 0$ is clear. Since $[S_n]$ converge to $[S]$, we obtain the general case by adding to S_n and S suitable combinations of α_i. Moreover, if S_n and S have continuous super-potentials, then since smooth forms are dense in $\mathcal{D}^0_{k-p+1}$, the extensions of $\mathcal{U}_{S_n}$ converge to the extension of $\mathcal{U}_S$ uniformly on $*$-bounded subsets of $\mathcal{D}^0_{k-p+1}$.

Proposition 3.2.8. *Let S be a current in $\mathcal{D}_p$ with continuous super-potentials. Then S_θ has continuous super-potentials and S_θ converges SP-uniformly to S when $\theta \to 0$. In particular, S can be approximated SP-uniformly by smooth forms in $\mathcal{D}_p$.*

Proof. Observe that the second assertion is deduced from the first one as in the proof of Theorem 2.4.4. We prove now the first assertion. When S is smooth, by Proposition 2.3.2, S_θ converges to S in the $\mathcal{C}^1$-topology and the result is clear. Adding to S a combination of α_i allows to assume that $[S] = 0$. Then, we also have $[S_\theta] = 0$ for every θ. We only have to consider $|\theta| \leq 1$ since S_θ does not depend on θ when $|\theta| \geq 1$. Let R be a current in $\mathcal{D}^0_{k-p+1}$ with $\|R\|_* \leq 1$. Let $\mathcal{U}_{S_\theta}$

denote the super-potential of S_θ. By Lemma 3.2.5, when R is smooth, we have $\mathcal{U}_{S_\theta}(R) = \mathcal{U}_S(R_\theta)$. Since R_θ depends continuously on R, $\mathcal{U}_{S_\theta}$ admits a continuous extension to $\mathcal{D}^0_{k-p+1}$ and the last identity holds for all R.

It remains to check that $\mathcal{U}_{S_\theta}$ converges SP-uniformly to $\mathcal{U}_S$. Recall that since $\mathcal{U}_S$ is continuous, if $\|R\|_*$ is bounded, we have $\mathcal{U}_S(R) \to 0$ when $\|R\|_{\mathcal{C}^{-2}} \to 0$. We also have $\mathcal{U}_{S_\theta}(R) - \mathcal{U}_S(R) = \mathcal{U}_S(R_\theta - R)$. When $\theta \to 0$ and $\|R\|_* \leq 1$, $\|R_\theta - R\|_*$ is bounded and by Lemma 2.4.3, $\|R_\theta - R\|_{\mathcal{C}^{-2}}$ tends to 0 uniformly on R. Therefore, $\mathcal{U}_S(R_\theta - R)$ tends to 0 uniformly on R with $\|R\|_* \leq 1$. The result follows. □

3.3 Intersection of currents

We will define the intersection of two currents such that at least one of them has a continuous super-potential. The theory of intersection is far from being complete but we will see that the following properties suffice in order to study the dynamics of automorphisms. We refer the reader to [8, 9, 31] for the theory of intersection with currents of bidegree $(1,1)$ and [22, 25] for the case of bidegree (p,p) on local setting or on homogeneous manifolds, see also [5, 32].

Let S be a current in $\mathcal{D}_p$ and S' a current in $\mathcal{D}_{p'}$ with $p+p' \leq k$. Assume that S has a continuous super-potential. We will define the intersection $S \wedge S'$ as a current in $\mathcal{D}_{p+p'}$. This wedge-product satisfies some continuity properties. Let $\mathcal{U}_S$ be the α-normalized super-potential of S and let $(a_1, \ldots, a_h)$ denote the coordinates of $[S]$ in the basis $[\alpha]$. Define for any test smooth real form Φ of bidegree $(k-p-p', k-p-p')$:

$$\langle S \wedge S', \Phi\rangle := \mathcal{U}_S(dd^c\Phi \wedge S') + \sum_{1\leq i\leq h} a_i \langle \alpha_i, \Phi \wedge S'\rangle.$$

Lemma 3.3.1. *Assume that S or S' is smooth. Then $S \wedge S'$ coincides with the usual wedge-product of S and S'.*

Proof. Assume that S' is smooth. Observe that $\Phi \wedge S'$ is a potential of $dd^c\Phi \wedge S'$. Define $m_i := \langle \alpha_i, \Phi \wedge S'\rangle$. Then $\Phi \wedge S' - \sum m_i \alpha_i^\vee$ is an α-normalized potential of $dd^c\Phi \wedge S'$. Therefore,

$$\begin{aligned}\mathcal{U}_S(dd^c\Phi \wedge S') + \sum a_i m_i &= \langle S, \Phi \wedge S'\rangle - \sum m_i \langle S, \alpha_i^\vee\rangle + \sum a_i m_i \\ &= \langle S, \Phi \wedge S'\rangle.\end{aligned}$$

This implies that $S \wedge S'$ coincides with the usual wedge-product of S and S'. The computation still holds when S is smooth but S' is singular. □

Theorem 3.3.2. *Let S be a current in $\mathcal{D}_p$ and S' a current in $\mathcal{D}_{p'}$ with $p+p' \leq k$. Assume that S has continuous super-potentials. Then $S \wedge S'$, defined as above, is a current in $\mathcal{D}_{p+p'}$ which depends linearly on S, S'. Moreover, we have*

$$[S \wedge S'] = [S] \smile [S'] \quad \textit{and} \quad \|S \wedge S'\|_* \leq c\|S\|_*\|S'\|_*$$

for some constant $c > 0$ independent of S, S'. Let S_n be currents in $\mathcal{D}_p$ with continuous super-potentials converging SP-uniformly to S and S'_n be currents converging in $\mathcal{D}_{p'}$ to S'. Then, $S_n \wedge S'_n$ converge in $\mathcal{D}_{p+p'}$ to $S \wedge S'$.

Proof. It is clear that $\langle S \wedge S', \Phi \rangle$ depends continuously on the smooth test form Φ. Hence, $S \wedge S'$ is a current. Clearly, this current depends linearly on S and S'. By definition, since $\mathcal{U}_S$ is continuous, $S \wedge S'$ depends continuously on S'. We deduce using Theorem 2.4.4 that $[S \wedge S'] = [S] \smile [S']$ since this identity holds for S' smooth. In order to estimate $\|S \wedge S'\|_*$, it is enough to assume that S' is smooth positive. Writing S as a difference of positive closed current, we see that $\|S \wedge S'\|_* \lesssim \|S\|_* \|S'\|_*$. We use here that the mass of a positive closed current depends only on its cohomology class. The last assertion of the theorem is deduced directly from the definition of $S \wedge S'$. □

Proposition 3.3.3. *Assume that S, S', S_n, S'_n have continuous super-potentials and that S_n, S'_n converge SP-uniformly to S, S' respectively. Then $S \wedge S'$ and $S_n \wedge S'_n$ have also continuous super-potentials and $S_n \wedge S'_n$ converge SP-uniformly to $S \wedge S'$.*

Proof. The proposition is clear when S, S_n are linear combinations of α_i. Subtracting from S and S_n suitable combinations of α_i allows to assume that $[S] = [S_n] = 0$. So, if Φ is a smooth test form we have by definition $\langle S \wedge S', \Phi \rangle = \mathcal{U}_S(S' \wedge dd^c \Phi)$. We deduce that if R is smooth then $\mathcal{U}_{S \wedge S'}(R) = \mathcal{U}_S(S' \wedge R)$. Since S and S' have continuous super-potentials, $\mathcal{U}_S(S' \wedge R)$ can be extended continuously to R in $\mathcal{D}^0_{k-p-p'+1}$. So, $S \wedge S'$ has a continuous super-potential and the identity $\mathcal{U}_{S \wedge S'}(R) = \mathcal{U}_S(S' \wedge R)$ holds for all R in $\mathcal{D}^0_{k-p-p'+1}$. In the same way, we prove that $S_n \wedge S'_n$ has a continuous super-potential and $\mathcal{U}_{S_n \wedge S'_n}(R) = \mathcal{U}_{S_n}(S'_n \wedge R)$. It is now clear that $S_n \wedge S'_n$ converge SP-uniformly to $S \wedge S'$. □

The following result shows that the wedge-product is commutative and associative. The first property allows to define $S' \wedge S := S \wedge S'$ when S has continuous super-potentials and S' is singular.

Proposition 3.3.4. *Let S_i, $i = 1, 2, 3$, be currents in $\mathcal{D}_{p_i}$. Assume that S_1 and S_2 have continuous super-potentials, then*

$$S_1 \wedge S_2 = S_2 \wedge S_1 \quad \textit{and} \quad (S_1 \wedge S_2) \wedge S_3 = S_1 \wedge (S_2 \wedge S_3).$$

Proof. The proposition is clear when S_1 and S_2 are smooth. The general case is deduced from this particular case using Propositions 3.2.8, 3.3.3 and Theorem 3.3.2. □

Remark 3.3.5. Assume that S and S' are positive currents. By Theorem 3.3.2, if S is SP-uniformly approximable by smooth positive closed (p, p)-forms, then $S \wedge S'$ is positive. This is also the case when S' can be approximated by positive smooth forms. In general, we don't know if $S \wedge S'$ is always positive when S or S' has continuous super-potentials.

3.4 Hölder super-potentials and moderate currents

Consider a current S in $\mathcal{D}_p$ with continuous super-potentials. Its super-potentials are defined on $\mathcal{D}^0_{k-p+1}$.

Definition 3.4.1. We say that S has *a Hölder continuous super-potential* if it admits a super-potential which is Hölder continuous on $*$-bounded subsets of $\mathcal{D}^0_{k-p+1}$ with respect to dist_l for some real number $l > 0$.

In order to prove that $\mathcal{U}_S$ is Hölder continuous, it is enough to show that $|\mathcal{U}_S(R)| \lesssim \|R\|^\lambda_{\mathcal{C}^{-l}}$ for $\|R\|_* \leq 1$ and for some constant $\lambda > 0$. By Proposition 2.2.1, the definition does not depend on the choice of l. One checks easily that the super-potentials of smooth forms are Hölder continuous. Hence, if S admits a Hölder continuous super-potential, all the super-potentials of S are Hölder continuous. In other words, this notion does not depend on the normalization of the super-potential.

Proposition 3.4.2. *Let S, S' be currents in $\mathcal{D}_p$ and $\mathcal{D}_{p'}$, $p + p' \leq k$, having Hölder continuous super-potentials. Then $S \wedge S'$ has a Hölder continuous super-potential.*

Proof. We can assume that $[S] = 0$ and $[S'] = 0$. Let $\mathcal{U}_S$, $\mathcal{U}_{S'}$ and $\mathcal{U}$ be the super-potentials of S, S' and $S \wedge S'$ respectively. So, for R in $\mathcal{D}^0_{k-p-p'+1}$ we have $\mathcal{U}(R) = \mathcal{U}_S(S' \wedge R)$. It is enough to prove for R in a $*$-bounded subset of $\mathcal{D}^0_{k-p-p'+1}$ that

$$\|S' \wedge R\|_{\mathcal{C}^{-4}} \lesssim \|R\|^\lambda_{\mathcal{C}^{-2}},$$

where $\lambda > 0$ is a constant. Using a regularization, we can assume that R is smooth. Let U' be a potential of S' and Φ a test form with $\|\Phi\|_{\mathcal{C}^4} \leq 1$. We have since $\mathcal{U}_{S'}$ is Hölder continuous and $\|dd^c\Phi\|_{\mathcal{C}^2}$ is bounded

$$\begin{aligned}
\|S' \wedge R\|_{\mathcal{C}^{-4}} &= \sup_\Phi |\langle S' \wedge R, \Phi\rangle| = \sup_\Phi |\langle U' \wedge R, dd^c\Phi\rangle| \\
&= \sup_\Phi |\langle U', R \wedge dd^c\Phi\rangle| = \sup_\Phi |\mathcal{U}_{S'}(R \wedge dd^c\Phi)| \\
&\lesssim \|R \wedge dd^c\Phi\|^\lambda_{\mathcal{C}^{-2}} \lesssim \|R\|^\lambda_{\mathcal{C}^{-2}}.
\end{aligned}$$

The result follows. □

Moderate currents and moderate measures were introduced in [17, 18]. With respect to test d.s.h. functions, moderate measures have the same regularity as the Lebesgue measure does.

Definition 3.4.3. A positive measure ν on X is *moderate* if there are constants $\lambda > 0$ and $A > 0$ such that

$$\langle \nu, e^{\lambda|\phi|}\rangle \leq A$$

for every d.s.h. function ϕ on X such that $\|\phi\|_{\mathrm{DSH}} \leq 1$. A measure is *moderate* if it is a difference of moderate positive measures. A positive closed (p,p)-current

S is *moderate* if its trace measure $S \wedge \omega^{k-p}$ is moderate and a current in $\mathcal{D}_p$ is *moderate* if it is a difference of moderate positive closed (p,p)-currents.

Proposition 3.4.4. *Let S be a positive closed (p,p)-current. Assume that S has a Hölder continuous super-potential. Then S is moderate.*

Proof. By Proposition 3.4.2, the trace measure of S has a Hölder continuous super-potential. Replacing S by its trace measure allows to assume that S is a positive measure, i.e. $p = k$. Let ϕ be a d.s.h. function with $\|\phi\|_{\rm DSH} \leq 1$. Define $\psi_M := \min(|\phi|, M) - \min(|\phi|, M-1)$ for $M \geq 1$. Observe that $0 \leq \psi_M \leq 1$ and that $\psi_M = 0$ on $\{|\phi| \leq M-1\}$, also ψ_M is larger than or equal to the characteristic function ρ_M of $\{|\phi| \geq M\}$. Moreover, the DSH-norm of ψ_M is bounded by a constant independent of ϕ and M, see e.g. [21]. We want to prove that $\langle S, e^{\lambda|\phi|}\rangle \leq A$ for some positive constants λ and A. So, it is enough to check that $\langle S, \rho_M\rangle \lesssim e^{-\lambda M}$ for some (other) positive constant λ. For this purpose, we will show that $\langle S, \psi_M\rangle \lesssim e^{-\lambda M}$.

By Proposition 2.2.3, the volume of the support of ψ_M is $\lesssim e^{-\lambda M}$ since it is contained in $\{|\phi| \geq M-1\}$. Therefore, the estimate $\langle S, \psi_M\rangle \lesssim e^{-\lambda M}$ is clear when S is a form with bounded $\|\cdot\|_\infty$-norm because $0 \leq \psi_M \leq 1$. Subtracting from S a smooth form allows to assume that $[S] = 0$ but we loose here the positivity of S. Recall that the super-potential $\mathcal{U}_S$ of S is Hölder continuous and that ψ_M has a bounded DSH-norm. We have for some constant $\lambda > 0$

$$\langle S, \psi_M\rangle = \mathcal{U}_S(dd^c\psi_M) \lesssim \|dd^c\psi_M\|_{\mathcal{C}^{-2}}^{\lambda}.$$

On the other hand, if Φ is a test form with $\|\Phi\|_{\mathcal{C}^2} \leq 1$ then

$$\|dd^c\psi_M\|_{\mathcal{C}^{-2}} = \sup_\Phi |\langle dd^c\psi_M, \Phi\rangle| = \sup_\Phi |\langle \psi_M, dd^c\Phi\rangle| \lesssim e^{-\lambda M},$$

where the last inequality follows from the above volume estimate of the support of ψ_M. This completes the proof. □

Proposition 3.4.5. *Let S be a positive closed (p,p)-current with Hölder continuous super-potentials on X. Assume that the manifold X is projective. Then the Hausdorff dimension of S is strictly larger than $2(k-p)$. More precisely, the trace measure $S \wedge \omega^{k-p}$ has no mass on sets of finite Hausdorff measure of dimension $2(k-p)+\epsilon$ for $\epsilon > 0$ small enough.*

We will need the following lemma where we use that X is projective.

Lemma 3.4.6. *Let $A > 0$ be a constant large enough and $r_0 > 0$ a constant small enough. If B_{r_0}, B_r are concentric balls of radius r_0, r respectively, $r \ll r_0$, then there is a positive smooth form Φ of bidegree $(k-p, k-p)$ supported in B_{r_0} with $\Phi \geq \omega^{k-p}$ on B_r and such that*

$$\|\Phi\| \leq Ar^{2k-2p+2}, \quad \|dd^c\Phi\|_* \leq Ar^{2k-2p} \quad \textit{and} \quad \|dd^c\Phi\|_{\mathcal{C}^{-1}} \leq Ar^{2k-2p+1}.$$

Proof. The case where X is the projective space $\mathbb{P}^k$ is proved in [25, Lemma 3.3.7]. We will deduce the lemma from this particular case. Since r_0 is small, we can choose a finite family of holomorphic maps from X onto $\mathbb{P}^k$ such that every ball of radius $3r_0$ is sent injectively to $\mathbb{P}^k$ by at least one of these maps. Let $\Pi : X \to \mathbb{P}^k$ be such a map corresponding to the ball B_{3r_0} with the same center as the considered balls B_r and B_{r_0}. Then, $\Pi(B_{r_0})$ contains a ball B' in $\mathbb{P}^k$ of radius $\gtrsim r_0$ and $\Pi(B_r)$ is contained in a ball B'' of radius $\lesssim r$. Let Ψ be a form satisfying the lemma for $\mathbb{P}^k$, B', B'' and for a fixed Kähler metric on $\mathbb{P}^k$. The choice of Π implies that the jacobian of $(\Pi_{|B_{2r_0}})^{-1}$ is bounded from below and from above by positive constants. Therefore, the form $\Phi := \Pi^*_{|B_{r_0}}(\Psi)$ is positive with support in B_{r_0}. It satisfies $\Phi \gtrsim \omega^{k-p}$ on B_r and $\|\Phi\| \lesssim r^{2k-2p+2}$, $\|dd^c\Phi\|_* \lesssim r^{2k-2p}$ on X. Multiplying Ψ by a constant allows to get $\Phi \geq \omega^{k-p}$ on B_r. Finally, it remains to check the inequality $\|dd^c\Phi\|_{\mathcal{C}^{-1}} \lesssim r^{2k-2p+1}$. We have to show that $\sup_\Omega |\langle dd^c\Phi, \Omega\rangle| \lesssim r^{2k-2p+1}$ for smooth test form Ω with $\|\Omega\|_{\mathcal{C}^1} \leq 1$. Since Φ is supported in B_{r_0}, it is enough to consider Ω with support in B_{2r_0}. In that case, the desired estimate is deduced from the analogous estimate for Ψ in $\mathbb{P}^k$. □

End of the proof of Proposition 3.4.5. Fix a constant $\epsilon > 0$ small enough. We only have to prove that $\int_{B_r} S\wedge\omega^{k-p} \lesssim r^{2k-2p+\epsilon}$ for r small, see e.g. [43]. Using the previous lemma, it suffices to check that $\langle S, \Phi\rangle \lesssim r^{2k-2p+\epsilon}$. The estimate is clear when S is smooth. Subtracting from S a smooth form allows to assume that $[S] = 0$ but we loose the positivity of S. Let $\mathcal{U}_S$ be the super-potential of S. Since $\mathcal{U}_S$ is Hölder continuous and $\|r^{2p-2k}dd^c\Phi\|_* \leq A$, we have

$$\begin{aligned}\langle S, \Phi\rangle &= \mathcal{U}_S(dd^c\Phi) = r^{2k-2p}\mathcal{U}_S(r^{2p-2k}dd^c\Phi)\\ &\lesssim r^{2k-2p}\|r^{2p-2k}dd^c\Phi\|^\epsilon_{\mathcal{C}^{-1}} \lesssim r^{2k-2p+\epsilon}.\end{aligned}$$

This implies the proposition. □

4 Dynamics of automorphisms

In this section, we study the dynamics of automorphisms on compact Kähler manifolds. The main dynamical objects (Green currents and equilibrium measure) were constructed by the authors in [20]. In [35], under some extra hypothesis, Guedj gives another construction of the Green current of some bidegree and of the equilibrium measure. Here, the theory of super-potentials allows us to obtain a new construction and to prove some fine properties of these dynamical objects.

4.1 Action on currents and on cohomology groups

We first give some basic properties of linear maps. Their proofs are left to the reader. Recall that a *Jordan block* $J_{\lambda,m}$ is a square complex matrix $(a_{ij})_{1\leq i,j\leq m}$

such that $a_{ij} = \lambda$ if $i = j$, $a_{ij} = 1$ if $j = i + 1$ and $a_{ij} = 0$ otherwise. If $\lambda \neq 0$, the entry of index $(1, m)$ of $J^n_{\lambda,m}$ is equal to $\binom{n}{m-1}\lambda^{n-m+1}$ when $n \geq m - 1$. This is the only entry of order $n^{m-1}|\lambda|^n$, the other ones have order at most equal to $n^{m-2}|\lambda|^n$. We have

$$\|J^n_{\lambda,m}\| \sim \binom{n}{m-1}|\lambda|^{n-m+1} \sim n^{m-1}|\lambda|^n.$$

The eigenspace of $J_{\lambda,m}$ associated to the unique eigenvalue λ is a complex line.

Consider a linear automorphism L of a real space $E \simeq \mathbb{R}^h$. We assume there is an open convex cone $\mathcal{K}$ in E which is salient, i.e. $\overline{\mathcal{K}} \cap -\overline{\mathcal{K}} = \{0\}$, and totally invariant by L, i.e. $L(\mathcal{K}) = \mathcal{K}$. For a fixed basis of E, L is associated to an invertible square matrix with real coefficients. One can extend L to an automorphism of $E^{\mathbb{C}} := E \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}^h$. Then, there is a complex basis of $E^{\mathbb{C}}$ such that the associated matrix of L is a Jordan matrix, i.e. a block diagonal matrix whose blocks are Jordan blocks. In other words, one can decompose $E^{\mathbb{C}}$ into direct sum of complex subspaces $E^{\mathbb{C}}_l$ which are invariant by L:

$$E^{\mathbb{C}} = \bigoplus_{1 \leq l \leq r} E^{\mathbb{C}}_l \quad \text{with} \quad \dim E^{\mathbb{C}}_l = m_l \quad \text{and} \quad \sum_{l=1}^{r} m_l = h,$$

such that the restriction L_l of L to $E^{\mathbb{C}}_l$ is defined by a Jordan block J_{λ_l,m_l}.

Up to a permutation of the $E^{\mathbb{C}}_l$, we can assume that the $(|\lambda_l|, m_l)$ are ordered so that either $|\lambda_l| > |\lambda_{l+1}|$ or $|\lambda_l| = |\lambda_{l+1}|$ and $m_l \geq m_{l+1}$ for every $1 \leq l \leq r - 1$. We say that J_{λ_l,m_l} is a *dominant* Jordan block if $(|\lambda_l|, m_l) = (|\lambda_1|, m_1)$ and in that case we say that λ_l is a *dominant eigenvalue* of L. Let ν be the integer such that $J_{\lambda_1,m_1}, \ldots, J_{\lambda_\nu,m_\nu}$ are the dominant Jordan blocks. The positive number $\lambda := |\lambda_1|$ is the *spectral radius* of L. The integer $m := m_1$ is called the *multiplicity of the spectral radius*. Since L preserves the salient open cone $\mathcal{K}$, λ is a dominant eigenvalue of L. Moreover, the Perron-Frobenius theorem implies that L admits an eigenvector in $\overline{\mathcal{K}}$ associated to the dominant eigenvalue λ. It is clear that $\|L^n\| \sim n^{m_1-1}\lambda^n$. Let $\widetilde{E}^{\mathbb{C}}_l$ be the hyperplane generated by the first $m_l - 1$ vectors of the basis of $E^{\mathbb{C}}_l$ associated to the Jordan form. We have $\|L^n v\| \sim n^{m_1-1}\lambda^n$ for any vector $v \notin \widetilde{E}^{\mathbb{C}}_1 \oplus \cdots \oplus \widetilde{E}^{\mathbb{C}}_\nu \oplus E^{\mathbb{C}}_{\nu+1} \oplus \cdots \oplus E^{\mathbb{C}}_r$, in particular for $v \in \mathcal{K}$.

Let $F^{\mathbb{C}}_l$ denote the eigenspace of L_l which is a complex line. We say that $F^{\mathbb{C}} := F^{\mathbb{C}}_1 \oplus \cdots \oplus F^{\mathbb{C}}_\nu$ is the *dominant eigenspace*. Define $H^{\mathbb{C}} := \oplus F^{\mathbb{C}}_l$ with $1 \leq l \leq \nu$ and $\lambda_l = \lambda$. This is the *strictly dominant eigenspace* of L. Define also $F := F^{\mathbb{C}} \cap E$ and $H := H^{\mathbb{C}} \cap E$. One can check that $F^{\mathbb{C}} = F \otimes_{\mathbb{R}} \mathbb{C}$ and $H^{\mathbb{C}} = H \otimes_R \mathbb{C}$. The previous spaces are invariant under L.

For any $1 \leq l \leq \nu$, there is a unique $\theta_l \in \mathbb{S} := \mathbb{R}/2\pi\mathbb{Z}$ such that $\lambda_l = \lambda \exp(i\theta_l)$. We say that $\theta := (\theta_1, \ldots, \theta_\nu) \in \mathbb{S}^\nu$ is the *dominant direction* of L. The dominant direction of L^n is equal to $n\theta$. Denote by Θ the closed subgroup of $\mathbb{S}^\nu$ generated by θ. It is a finite union of real tori. The orbit of each point $\theta' \in \Theta$ under the translation $\theta' \mapsto \theta' + \theta$ is dense in Θ. If $\lambda_l = \lambda$ for every $1 \leq l \leq \nu$, we have $F^{\mathbb{C}} = H^{\mathbb{C}}$, $\theta = 0$ and $\Theta = \{0\}$. Define

$$\widehat{L}_N := \frac{1}{N} \sum_{n=1}^{N} \frac{L^n}{n^{m-1}\lambda^n}.$$

We have the following proposition, see also [20].

Proposition 4.1.1. *The sequence $(\widehat{L}_N)$ converges to a surjective real linear map $\widehat{L}_\infty : E \to H$. Let (n_i) be an increasing sequence of integers. Then $(n_i^{1-m}\lambda^{-n_i}L^{n_i})$ converges if and only if $(n_i\theta)$ converges. Moreover, any limit of $(n^{1-m}\lambda^{-n}L^n)$ is a surjective real linear map from E to F.*

Note that surjective linear maps are open and the image of $\mathcal{K}$ by such a map is an open convex cone.

We will apply the previous result to the action of a holomorphic map on cohomology groups. Let f be an automorphism on a compact Kähler manifold (X, ω) of dimension k. The pull-back operator f^* acts on smooth forms and on currents. It commutes with ∂, $\overline{\partial}$ and preserves positivity. Therefore, f^* acts as a linear automorphism on $H^{q,q}(X, \mathbb{R})$. The operator push-forward f_* is defined in the same way. It coincides with the pull-back $(f^{-1})^*$ by f^{-1}.

Recall that *the dynamical degree of order* q of f is the spectral radius of f^* acting on $H^{q,q}(X, \mathbb{R})$. Let us denote by $d_q(f)$ (or d_q if there is no confusion) this degree. We have $d_q(f^n) = d_q(f)^n$ for $n \geq 1$ and $d_0(f) = d_k(f) = 1$. An inequality due to Khovanskii, Teissier and Gromov [33] implies that the function $q \mapsto \log d_q$ is concave on $0 \leq q \leq k$, see also [35]. In particular, there are two integers p and p' with $0 \leq p \leq p' \leq k$ such that

$$1 = d_0 < \cdots < d_p = \cdots = d_{p'} > \cdots > d_k = 1.$$

By Gromov and Yomdin [34, 48], the dynamical degrees are related to the topological entropy $h_t(f)$ of f by the formula $h_t(f) = \max_q \log d_q$, see also [19] for a more general context.

Let $\mathcal{K}$ be the convex cone of the classes in $H^{q,q}(X, \mathbb{R})$ associated to strictly positive closed (q, q)-forms. Then $\mathcal{K}$ is salient and totally invariant by f^*. Hence, we can apply Proposition 4.1.1 to f^* (one can also apply it to the cone of the classes associated to strictly positive closed (q, q)-currents). Recall that the bilinear form $\smile$ on $H^{q,q}(X, \mathbb{R}) \times H^{k-q,k-q}(X, \mathbb{R})$ given by

$$([\beta], [\beta']) \mapsto [\beta] \smile [\beta'] := \int_X \beta \wedge \beta'$$

is non-degenerate. Moreover, we have $f^*[\beta] \smile [\beta'] = [\beta] \smile f_*[\beta']$. So, if we consider two basis of $H^{q,q}(X, \mathbb{R})$ and of $H^{k-q,k-q}(X, \mathbb{R})$ which are dual with respect to $\smile$, then f^* acting on $H^{q,q}(X, \mathbb{R})$ and f_* acting on $H^{k-q,k-q}(X, \mathbb{R})$ are given by the same matrix. Therefore, these operators have the same spectral radius $d_q(f) = d_{k-q}(f^{-1})$ with the same multiplicity m.

Lemma 4.1.2. *If S is a current in $\mathcal{D}_q$ then*

$$\|(f^n)^*S\|_* \leq \kappa n^{m-1} d_q^n \|S\|_*$$

where $\kappa > 0$ is a constant independent of S. Moreover, if S is a strictly positive current, we have $\|(f^n)^*(S)\| \sim n^{m-1}d_q^n$.

Proof. We can assume that S is positive. The mass of a positive closed current can be computed cohomologically. Therefore,

$$\|(f^n)^*S\| = (f^n)^*[S] \smile [\omega^{k-q}] \lesssim \|(f^n)^*\|\|[S]\| \lesssim n^{m-1}d_q^n\|S\|.$$

This gives the first part of the lemma. For the second one, if S is strictly positive then $[S]$ is in the interior of the cone of the classes of positive closed currents. We deduce from the above discussion on the linear operator L that $\|(f^n)^*[S]\| \sim n^{m-1}d_q^n$. The result follows. □

Note that the previous lemma allows to compute the dynamical degrees using the following formula

$$d_q(f) = \lim_{n\to\infty} \Big[\int_X (f^n)^*\omega^q \wedge \omega^{k-q} \Big]^{1/n} = \lim_{n\to\infty} \Big[\int_X \omega^q \wedge (f^n)_*\omega^{k-q} \Big]^{1/n}.$$

4.2 Construction of Green currents

In this section, we give a new construction of the Green currents using the super-potentials. This approach permits to establish some new properties of the Green currents. The result can be extended to open non-invertible maps, see [23] for the pull-back operator on currents by non-invertible maps. We use here the notations introduced in Section 4.1.

Theorem 4.2.1. *Let f be a holomorphic automorphism on a compact Kähler manifold (X, ω). Let d_s be the dynamical degrees of f and q an integer such that $d_{q-1} < d_q$. Let F (resp. H) denote the real dominant (resp. strictly dominant) subspace associated to the operator f^* on $H^{q,q}(X,\mathbb{R})$. Then each class c of F can be represented by a current T_c in $\mathcal{D}_q$ with Hölder continuous super-potentials which depends linearly on c and satisfies $f^*(T_c) = T_{f^*(c)}$. In particular, we have $f^*(T_c) = d_qT_c$ for $c \in H$. The set of the classes c in F (resp. in H) with T_c positive is a closed convex cone with non-empty interior.*

The cone of positive closed currents T_c with $c \in H$ is a closed cone of finite dimension. We say that T_c is *a Green current of order q* of f if T_c is a non-zero positif current (this implies that $c \neq 0$). By Proposition 3.4.4, the Green currents are moderate. We will see that Green currents are the only positive closed currents in their cohomology classes.

Consider now the action of f^* on $H^{q,q}(X,\mathbb{R})$ as described in Section 4.1. Let m denote the multiplicity of its spectral radius d_q.

Proposition 4.2.2. *Let S be a current in $\mathcal{D}_q$ with a continuous super-potential. Let (n_i) be an increasing sequence of integers. Assume that $n_i^{1-m} d_q^{-n_i} (f^{n_i})^*[S]$ converge to some class c in $H^{q,q}(X, \mathbb{R})$. Then $n_i^{1-m} d_q^{-n_i} (f^{n_i})^*(S)$ converge SP-uniformly to a current T_c in $\mathcal{D}_q$ which depends only on c.*

Let $\alpha = \{\alpha_1, \ldots, \alpha_h\}$ be a family of smooth closed real (q,q)-forms such that $[\alpha] = \{[\alpha_1], \ldots, [\alpha_h]\}$ is a basis of $H^{q,q}(X, \mathbb{R})$ where h is the dimension of $H^{q,q}(X, \mathbb{R})$. In what follows, we consider the super-potentials normalized by α as in Section 3. Let M denote the $h \times h$ matrix whose column of index j is given by the coordinates of $f^*[\alpha_j]$ with respect to the basis $[\alpha]$. Let $\mathcal{U}_j$ denote the super-potential of $f^*(\alpha_j)$ and define $\mathcal{U} := (\mathcal{U}_1, \ldots, \mathcal{U}_h)$. Let $A = {}^t(a_1, \ldots, a_h)$ denote the coordinates of $[S]$ in the basis $[\alpha]$ and $\mathcal{U}_S$, $\mathcal{U}_{S_n}$ the super-potentials of S and of $S_n := (f^n)^* S$ respectively. Denote also by Λ the operator f_* acting on $\mathcal{D}^0_{k-q+1}$.

Lemma 4.2.3. *We have*

$$\mathcal{U}_{S_n} = \sum_{l=0}^{n-1} (\mathcal{U} \circ \Lambda^l) M^{n-l-1} A + \mathcal{U}_S \circ \Lambda^n.$$

Proof. The proof is by induction. For $n = 0$, we have $S_0 = S$ and the lemma is clear. Assume the lemma for n. We show it for $n+1$. Let R be a smooth form in $\mathcal{D}^0_{k-q+1}$ and U a smooth potential of R normalized by α. So, $f_*(U)$ is a potential of $\Lambda(R)$ but it is not normalized. Let $\alpha^\vee = \{\alpha_1^\vee, \ldots, \alpha_h^\vee\}$ be a family of smooth real closed forms such that $[\alpha^\vee]$ is the basis of $H^{k-q,k-q}(X, \mathbb{R})$ which is dual to $[\alpha]$ with respect to $\smile$. Define

$$b_j := \langle \alpha_j, f_*(U) \rangle = \langle f^*(\alpha_j), U \rangle = \mathcal{U}_j(R)$$

and

$$b := {}^t(b_1, \ldots, b_m) = {}^t\mathcal{U}(R).$$

Then $U' := f_*(U) - \alpha^\vee b$ is a potential of $\Lambda(R)$ normalized by α. We obtain using the induction hypothesis

$$\begin{aligned}
\mathcal{U}_{S_{n+1}}(R) &= \langle S_{n+1}, U \rangle = \langle f^*(S_n), U \rangle = \langle S_n, f_*(U) \rangle \\
&= \langle S_n, U' \rangle + \langle S_n, \alpha^\vee b \rangle = \mathcal{U}_{S_n}(\Lambda(R)) + \langle S_n, \alpha^\vee b \rangle \\
&= \sum_{l=1}^{n} \mathcal{U}(\Lambda^l(R)) M^{n-l} A + \mathcal{U}_S(\Lambda^{n+1}(R)) + \langle S_n, \alpha^\vee b \rangle.
\end{aligned}$$

We only have to check that the last integral satisfies

$$\langle S_n, \alpha^\vee b \rangle = \mathcal{U}(R) M^n A.$$

Observe that the integral $\langle S_n, \alpha^\vee b\rangle$ can be computed cohomologically. Since S is cohomologous to αA, by definition of M, $S_n = (f^n)^* S$ is cohomologous to $\alpha M^n A$. Hence

$$\langle S_n, \alpha^\vee b\rangle = \langle \alpha M^n A, \alpha^\vee b\rangle = {}^t b M^n A = \mathcal{U}(R) M^n A.$$

This completes the proof. □

End of the proof of Proposition 4.2.2. By Proposition 4.1.1 the limit c of $n_i^{1-m} d_q^{-n_i} (f^{n_i})[S]$ is a class in F. Write $c = {}^t(c_1, \ldots, c_h)$ with respect to the basis $[\alpha]$. Then, $n_i^{1-m} d_q^{-n_i} M^{n_i} A$ converges to c.

By Lemma 4.2.3, the super-potential $\mathcal{U}_{n_i}$ of $n_i^{1-m} d_q^{-n_i} (f^{n_i})^*(S)$ is equal to

$$\begin{aligned}\mathcal{U}_{n_i} &= n_i^{1-m} d_q^{-n_i} \Big[\sum_{l=0}^{n_i-1} (\mathcal{U} \circ \Lambda^l) M^{n_i-l-1} A + \mathcal{U}_S \circ \Lambda^{n_i} \Big] \\ &= \sum_{l=0}^{n_i-1} (\mathcal{U} \circ \Lambda^l) \frac{M^{n_i-l-1} A}{n_i^{m-1} d_q^{n_i}} + n_i^{1-m} d_q^{-n_i} \mathcal{U}_S \circ \Lambda^{n_i}. \qquad (4.1)\end{aligned}$$

Since $\mathcal{U}_S$ is continuous, we have

$$|\mathcal{U}_S \circ \Lambda^n(R)| \lesssim \|\Lambda^n(R)\|_* \lesssim \delta^n \|R\|_*, \qquad (4.2)$$

where $d_{q-1} < \delta < d_q$ is a fixed constant. It follows that the last term in (4.1) tends uniformly to 0 on $*$-bounded sets of R.

Recall that $\|M^n\| \sim n^{m-1} d_q^n$. Analogous estimates as in (4.2) for $\mathcal{U}_j$ imply that

$$\Big\| \frac{M^{n_i-l-1} A}{n_i^{m-1} d_q^{n_i}} \Big\| \lesssim d_q^{-l} \quad \text{and} \quad \Big| (\mathcal{U}_j \circ \Lambda^l) \frac{M^{n_i-l-1} A}{n_i^{m-1} d_q^{n_i}} \Big| \lesssim \delta^l d_q^{-l}. \qquad (4.3)$$

Since $\sum_{l\geq 0} \delta^l d_q^{-l}$ converges, we can apply the Lebesgue convergence theorem for the sum in (4.1). We obtain the uniform convergence on $*$-bounded sets:

$$\lim_{i\to\infty} \mathcal{U}_{n_i} = \sum_{l\geq 0} (\mathcal{U} \circ \Lambda^l) M^{-l-1} c.$$

The last series converge because (4.3) implies that $\|M^{-l-1} c\| \lesssim d_q^{-l}$. One can also obtain this inequality using the fact that c is a vector in F and that the matrix of $f^*_{|F}$ is conjugate to a diagonal matrix whose eigenvalues are of modulus d_q.

Hence, the sequence $(n_i^{1-m} d_q^{-n_i} (f^{n_i})^*(S))$ converges to some current T_c. Moreover, the last series defines a super-potential $\mathcal{U}_{T_c}$ of T_c and $\mathcal{U}_{n_i}$ converge to $\mathcal{U}_{T_c}$ uniformly on $*$-bounded sets of R. Hence, the convergence of $(n_i^{1-m} d_q^{-n_i} (f^{n_i})^*(S))$ is SP-uniform. Since $\mathcal{U}_{T_c}$ depends only on the class c, by Proposition 3.2.3, T_c depends only on this class. □

Proposition 4.2.4. *Let (n_i) be an increasing sequence such that $(n_i^{1-m} d_q^{-n_i} (f^{n_i})^*)$ converges on $H^{q,q}(X, \mathbb{R})$. Then for any class $c \in F$ there is a smooth form S in $\mathcal{D}_q$ such that $n_i^{1-m} d_q^{-n_i} (f^{n_i})^*(S)$ converge SP-uniformly to T_c.*

Proof. By Proposition 4.1.1, one can find a smooth form S in $\mathcal{D}_q$ such that $n_i^{1-m} d_q^{-n_i} (f^{n_i})^*[S]$ converge to c. It is enough to apply Proposition 4.2.2. □

Lemma 4.2.5. *The current T_c in Proposition 4.2.2 has Hölder continuous potential.*

Proof. We follow the approach in [20] and [25]. Let R be a smooth form in $\mathcal{D}^0_{k-q+1}$ such that $\|R\|_* \leq 1$. Since $f^*(\alpha_j)$ is smooth, its super-potential $\mathcal{U}_j$ is Lipschitz in the sense that $\|\mathcal{U}_j(R)\| \leq \kappa \|R\|_{\mathcal{C}^{-1}}$ with $\kappa > 1$ independent of R. By definition of $\|\cdot\|_{\mathcal{C}^{-1}}$, we also have $\|\Lambda(R)\|_{\mathcal{C}^{-1}} \leq \kappa \|R\|_{\mathcal{C}^{-1}}$ for some constant $\kappa > 1$.

Let δ be a constant as above with $d_{q-1} < \delta < d_q$. Define $\rho := \delta d_q^{-1}$, $\widetilde{\Lambda} := \delta^{-1}\Lambda$, $\lambda := -\log \rho (\log \kappa - \log \rho)^{-1}$ and N_0 the integer part of $(\lambda - 1) \log \|R\|_{\mathcal{C}^{-1}} (\log \kappa)^{-1}$. Then the sequence $(\widetilde{\Lambda}^l)_{l\geq 0}$ is bounded with respect to the $\|\cdot\|_*$-norm. For $\|R\|_{\mathcal{C}^{-1}}$ small enough, we have since $\|M^{-l-1}c\| \lesssim d_q^{-l}$

$$
\begin{aligned}
|\sum_{l\geq 0} (\mathcal{U} \circ \Lambda^l(R)) M^{-l-1} c| &\lesssim \sum_{l=0}^{N_0} \rho^l \|\mathcal{U} \circ \widetilde{\Lambda}^l(R)\| + \sum_{l>N_0} \rho^l \|\mathcal{U} \circ \widetilde{\Lambda}^l(R)\| \\
&\lesssim \Big(\sum_{l=0}^{N_0} \rho^l \kappa^l\Big) \|R\|_{\mathcal{C}^{-1}} + \sum_{l>N_0} \rho^l \\
&\lesssim \kappa^{N_0} \|R\|_{\mathcal{C}^{-1}} + \rho^{N_0} \lesssim \|R\|^{\lambda}_{\mathcal{C}^{-1}}.
\end{aligned}
$$

Therefore, $\sum_{l\geq 0} (\mathcal{U} \circ \Lambda^l) M^{-l-1} c$ is λ-Hölder continuous with respect to dist_{-1}. □

End of the proof of Theorem 4.2.1. By Proposition 4.2.4, T_c depends linearly on c because it depends linearly on S. By Lemma 4.2.5, T_c has Hölder continuous super-potentials. Observe that if $n_i^{1-m} d_q^{-n_i} (f^{n_i})^*[S]$ converge to c then $n_i^{1-m} d_q^{-n_i} (f^{n_i})^*[f^*(S)]$ converge to $f^*(c)$. Applying Proposition 4.2.2 to S and to $f^*(S)$ yields

$$
f^*(T_c) = f^*\Big(\lim_{i\to\infty} n_i^{1-m} d_q^{-n_i} (f^{n_i})^*(S)\Big) = \lim_{i\to\infty} n_i^{1-m} d_q^{-n_i} (f^{n_i})^* \big(f^*(S)\big) = T_{f^*(c)}.
$$

If c is in H, we have $f^*(c) = d_q c$. Hence, $f^*(T_c) = d_q T_c$.

We deduce easily from the linear dependence of T_c on c that the cone $\mathcal{C}_F$ (resp. $\mathcal{C}_H$) of the classes c in F (resp. in H) with T_c positive is convex and closed. It remains to prove that they have non-empty interior. Observe that $\mathcal{C}_F$ contains the classes c associated to S smooth strictly positive and that the cone $\mathcal{K}$ in $H^{q,q}(X,\mathbb{R})$ of such forms S is open. By Proposition 4.1.1, any limit L_∞ of $(n^{1-m} d_q^{-n} (f^n)^*)$ is an open map from $H^{q,q}(X,\mathbb{R})$ to F. Hence, $\mathcal{C}_F$ contains the cone $L_\infty(\mathcal{K})$ which is open in F.

Consider as in Section 4.1 the operators

$$
\widehat{L}_N := \frac{1}{N} \sum_{n=1}^{N} n^{1-m} d_q^{-n} (f^n)^*
$$

on $H^{q,q}(X,\mathbb{R})$ and $\widehat{L}_\infty$ the limit of this sequence which is an open map from $H^{q,q}(X,\mathbb{R})$ to H. Observe that any class in $\widehat{L}_\infty(\mathcal{K})$ belongs to the closed convex cone generated by the $L_\infty(\mathcal{K})$. Hence, $\mathcal{C}_F$ contains $\widehat{L}_\infty(\mathcal{K})$. We deduce that $\mathcal{C}_H$, which is equal to $\mathcal{C}_F\cap H$, contains the open cone $\widehat{L}_\infty(\mathcal{K})$ of H. This completes the proof of Theorem 4.2.1. □

Proposition 4.2.6. *Let S be a current in $\mathcal{D}_q$ with continuous super-potentials. Let $\widehat{L}_N$ be as above and $\widehat{c}\in H$ the limit of the sequence $(\widehat{L}_N[S])$. Then $\widehat{L}_N(S)$ converge SP-uniformly to the current $T_{\widehat{c}}$. Moreover, any current $T_{\widehat{c}}$ with $\widehat{c}\in H$ can be obtained as the limit of $(\widehat{L}_N[S])$ for some S smooth in $\mathcal{D}_q$.*

Proof. Proposition 4.2.2 implies that any limit T of $\widehat{L}_N(S)$ belongs to the space generated by the T_c with $c\in F$. Hence, T is equal to one of the current T_c. On the other hand, T is a current in the class $\widehat{c}$. We deduce that $c=\widehat{c}$ and that $\widehat{L}_N(S)$ converge to $T_{\widehat{c}}$. The main point here is to show that the convergence is SP-uniform. We follows the proof of Proposition 4.2.2.

By Lemma 4.2.3 the super-potential $\mathcal{U}_{\widehat{L}_N(S)}$ of $\widehat{L}_N(S)$ is equal to

$$\begin{aligned}\mathcal{U}_{\widehat{L}_N(S)} &= \frac{1}{N}\sum_{n=1}^{N} n^{1-m}d_q^{-n}\Big[\sum_{l=0}^{n-1}(\mathcal{U}\circ\Lambda^l)M^{n-l-1}A+\mathcal{U}_S\circ\Lambda^n\Big]\\ &= \sum_{l=0}^{N-1} d_q^{-l-1}(\mathcal{U}\circ\Lambda^l)\Big[\frac{1}{N}\sum_{n=0}^{N-l-1}(n+l+1)^{1-m}d_q^{-n}M^nA\Big]+ \qquad (4.4)\\ &\quad +\frac{1}{N}\sum_{n=1}^{N} n^{1-m}d_q^{-n}\mathcal{U}_S\circ\Lambda^n. \qquad (4.5)\end{aligned}$$

Since $\mathcal{U}_S$ is continuous, the quantity in (4.5) tends to 0 uniformly on $*$-bounded sets. By Proposition 4.1.1, the term in the brackets in (4.4) converges to the vector of coordinates equal to the coordinates of $\widehat{c}$ in the basis $[\alpha]$. Denote also by $\widehat{c}$ this vector. We deduce that the expression in (4.4) converges uniformly on $*$-bounded sets to

$$\mathcal{U}_{T_{\widehat{c}}} := \sum_{l\geq 0} d_q^{-l-1}(\mathcal{U}\circ\Lambda^l)\widehat{c},$$

which defines a super-potential of $T_{\widehat{c}}$. Hence, the convergence of $\widehat{L}_N(S)$ is SP-uniform.

The last assertion of the proposition is deduced from the surjectivity of the map $\widehat{L}_\infty$ in Proposition 4.1.1. □

4.3 Uniqueness of Green currents and equidistribution

In this section, we will prove the uniqueness of the Green currents in their cohomology classes. We have the following general result.

Theorem 4.3.1. *Let f be a holomorphic automorphism of a compact Kähler manifold (X, ω) and d_s the dynamical degrees of f. Let V be a subspace of $H^{q,q}(X, \mathbb{R})$ invariant by f^*. Assume that $d_q > d_{q-1}$ and that all the (real and complex) eigenvalues of the restriction of f^* to V are of modulus strictly larger than d_{q-1}. Then each class in V contains at most one positive closed (q, q)-current.*

Proof. Let S and S' be positive closed currents in the same class in V. Define $\lambda_n := \|(f^n)_*(S)\|^{-1} = \|(f^n)_*(S')\|^{-1}$, $S_n := \lambda_n (f^n)_*(S)$ and $S'_n := \lambda_n (f^n)_*(S')$. The currents S_n, S'_n are of mass 1. We have $S = \lambda_n^{-1}(f^n)^*(S_n)$ and $S' = \lambda_n^{-1}(f^n)^*(S'_n)$. Let $\delta_1 > \delta_2 > d_{q-1}$ be constants such that the eigenvalues of $f^*_{|V}$ have modulus strictly larger than δ_1. We deduce that the eigenvalues of $f_{*|V}$ have modulus strictly smaller than δ_1^{-1}. Therefore, Lemma 4.1.2 applied to f_* implies that $\lambda_n \gtrsim \delta_1^n$. Let $\mathcal{U}_S$, $\mathcal{U}_{S'}$, $\mathcal{U}_{S_n}$, $\mathcal{U}_{S'_n}$ denote the super-potentials of S, S', S_n and S'_n respectively.

Assume that $S \neq S'$. Proposition 3.2.3 implies that $\mathcal{U}_S \neq \mathcal{U}_{S'}$. Then, there is a smooth form R in $\mathcal{D}^0_{k-q+1}$ such that $\mathcal{U}_S(R) - \mathcal{U}_{S'}(R) \neq 0$. If we multiply R by a constant, we can assume that $\mathcal{U}_S(R) - \mathcal{U}_{S'}(R) = 1$. Since S and S' are cohomologous, they have the same coordinates A in the basis $[\alpha]$. By Lemma 4.2.3, we have

$$\begin{aligned}\mathcal{U}_{S_n}((f^n)_*R) - \mathcal{U}_{S'_n}((f^n)_*R) &= \mathcal{U}_{(f^n)^*S_n}(R) - \mathcal{U}_{(f^n)^*S'_n}(R)\\ &= \lambda_n \mathcal{U}_S(R) - \lambda_n \mathcal{U}_{S'}(R)\\ &= \lambda_n.\end{aligned}$$

Define $R_n := \gamma_n^{-1}(f^n)_*(R)$ where γ_n is the norm of $(f^n)^*$ acting on $H^{q-1,q-1}(X, \mathbb{R})$. Recall that γ_n is also the norm of $(f^n)_*$ acting on $H^{k-q+1,k-q+1}(X, \mathbb{R})$. We have $\lim_{n\to\infty} \gamma_n^{1/n} = d_{q-1} < \delta_2$. Moreover, $\|R_n\|_*$ is bounded by a constant independent of n. We have for n large enough

$$\mathcal{U}_{S_n}(R_n) - \mathcal{U}_{S'_n}(R_n) = \lambda_n \gamma_n^{-1} \geq 2\delta_1^n \delta_2^{-n}.$$

It follows that either $|\mathcal{U}_{S_n}(R_n)| \geq \delta_1^n \delta_2^{-n}$ or $|\mathcal{U}_{S'_n}(R_n)| \geq \delta_1^n \delta_2^{-n}$. On the other hand, if κ is a fixed constant large enough, we have $\|f_*(R)\|_{\mathcal{C}^1} \leq \kappa \|R\|_{\mathcal{C}^1}$ since f is an automorphism, and by induction

$$\|R_n\|_{\mathcal{C}^1} = \gamma_n^{-1} \|(f^n)_*(R)\|_{\mathcal{C}^1} \lesssim \widetilde{\kappa}^n,$$

for some constant $\widetilde{\kappa}$. Theorem 3.2.6, applied to S_n and S'_n, implies that $\delta_1^n \delta_2^{-n} \lesssim n$. This is a contradiction because $\delta_1 > \delta_2$. □

Observe that in Theorem 4.3.1, by linearity, each class of V contains at most one current $T = T^+ - T^-$ with T^+, T^- positive closed and $[T^+]$, $[T^-]$ in V. We apply this theorem to V the maximal subspace of $H^{q,q}(X, \mathbb{R})$ where the eigenvalues of f^* are of modulus d_q. We obtain the following corollaries.

Corollary 4.3.2. *Let f, d_s and q be as in Theorem 4.2.1. Then the Green (q,q)-currents of f are the unique positive closed currents in their cohomology classes. They are the only non-zero positive closed (q,q)-currents which are invariant by $d_q^{-1}f^*$, i.e. satisfying the equation $d_q^{-1}f^*(T) = T$.*

Corollary 4.3.3. *Let f, d_s and q be as in Theorem 4.2.1. Let T be a Green (q,q)-current of f and $\mathcal{C}_T$ the set of positive closed (q,q)-currents S such that $S \leq cT$ for some constant $c > 0$. Then $\mathcal{C}_T$ is a salient convex closed cone of finite dimension. Moreover, each current in $\mathcal{C}_T$ is the unique positive closed current in its cohomology class.*

Proof. It is clear that $\mathcal{C}_T$ is a convex cone. It is salient since the cone of of all positive closed (q,q)-currents is salient. Let E^+ denote the cone of the classes of currents S in $\mathcal{C}_T$ and E the space generated by E^+. Then E^+ is convex and salient since it is contained in the cone of the classes of positive closed currents. Since T is a Green current, it is invariant by $d_q^{-1}f^*$ and $d_q f_*$. If v is a vector in E^+, then by definition of E^+, $\|(f^n)_*v\| \lesssim d_q^{-n}$. Therefore, the eigenvalues of f_* restricted to E are of modulus at most equal to d_q^{-1}. We deduce that all eigenvalues of f^* have modulus equal to d_q. By Theorem 4.3.1, S is the only positive closed current in $[S]$. Moreover, in $\overline{E}^+$, the current S depends linearly on its class. We deduce from the correspondence $S \leftrightarrow [S]$ and the definition of E^+ that $\overline{E}^+ = E^+$. So, $\mathcal{C}_T$ is closed. □

The following results can be applied to the currents of integration on subvarieties of pure codimension q of X, and give equidistribution properties of their images by f^{-n}.

Corollary 4.3.4. *Let f, d_s and q be as in Theorem 4.2.1. Let m denote the multiplicity of the spectral radius of f^* on $H^{q,q}(X,\mathbb{R})$. Let (S_i) be a sequence of positive closed (q,q)-currents. If (n_i) is an increasing sequence of integers such that $n_i^{1-m}d_q^{-n_i}(f^{n_i})^*[S_i]$ converge in $H^{q,q}(X,\mathbb{R})$, then $n_i^{1-m}d_q^{-n_i}(f^{n_i})^*(S_i)$ converge either to 0 or to a Green (q,q)-current.*

Proof. Let c denote the limit of $n_i^{1-m}d_p^{-n_i}(f^{n_i})^*[S_i]$. Observe that the sequence of $n_i^{1-m}d_p^{-n_i}(f^{n_i})^*[S_i]$ is bounded. Hence, the currents $n_i^{1-m}d_q^{-n_i}(f^{n_i})^*(S_i)$ have mass bounded by a constant independent of i. Then, we can extract convergence subsequences. All the limit currents are in the same class c of F. By Theorem 4.3.1, they are equal. This implies the result. □

Corollary 4.3.5. *Let f, d_s, q and m be as in Corollary 4.3.4. Let S be a positive closed (q,q)-current on X. Then the sequence*

$$\frac{1}{N}\sum_{n=1}^{N} n^{1-m}d_q^{-n}(f^n)^*(S)$$

converges either to 0 or to a Green (q,q)-current of f.

Proof. It is enough to apply Theorem 4.3.1 and to observe that by Proposition 4.1.1, the sequence

$$\frac{1}{N}\sum_{n=1}^{N} n^{1-m} d_q^{-n} (f^n)^*[S]$$

converges to a class in H. Note that when all the dominant eigenvalues of f^* on $H^{q,q}(X,\mathbb{R})$ are equal to d_q (i.e. real positive), then $n^{1-m}d_q^{-n}(f^n)^*[S]$ converges to a class c in H. Therefore, $n^{1-m}d_q^{-n}(f^n)^*(S)$ converge to 0 if $c = 0$ or to a Green current otherwise. □

4.4 Equilibrium measure, mixing and hyperbolicity

In this section, we assume that f admits a dynamical degree d_p strictly larger than the other ones. We have

$$1 = d_0 < \cdots < d_p > \cdots > d_k = 1.$$

Then, we can construct Green (q,q)-currents of f for $1 \leq q \leq p$ and Green (q,q)-currents associated to f^{-1} for $1 \leq q \leq k-p$.

If T^+ is a Green (p,p)-current of f and T^- is a Green $(k-p,k-p)$-current associated to f^{-1}, then as it is noticed in [20], we can define the intersection $T^+ \wedge T^-$, see also Section 3.3. This gives an invariant measure. However, we cannot prove that this measure does not vanish. We introduced in [20] another construction which always gives an ergodic probability measure[5]. This measure is the intersection of a Green current T^+ with $(1,1)$-currents with Hölder continuous potentials. The main result in [17] implies that the measure is moderate. Here is a criterion for the non-vanishing of $T^+ \wedge T^-$, see also [35].

Proposition 4.4.1. *If f is as above, then the following properties are equivalent*

1. *There is a Green (p,p)-current T^+ of f and a Green $(k-p,k-p)$-current T^- of f^{-1} such that $T^+ \wedge T^-$ is a positive non-zero measure.*

2. *The spectral radius of f^* on $H^{p,p}(X,\mathbb{R})$ is of multiplicity* 1.

3. *The spectral radius of f_* on $H^{k-p,k-p}(X,\mathbb{R})$ is of multiplicity* 1.

Proof. To say that the multiplicity of the spectral radius is 1 means that the Jordan blocks associated to the eigenvalues of maximal modulus are reduced to these eigenvalues. As it is showed in Section 4.1, if we consider a basis of $H^{p,p}(X,\mathbb{R})$ and $H^{k-p,k-p}(X,\mathbb{R})$ which are dual with respect to the cup-product $\smile$, then f^* acting on $H^{p,p}(X,\mathbb{R})$ and f_* acting on $H^{k-p,k-p}(X,\mathbb{R})$ are given by the same matrix. Therefore, properties 2 and 3 are equivalent.

[5] there is a slip at the end of [20, p.310]; the measure that we constructed is only ergodic, almost mixing and mixing when the dominant eigenvalues of f^* on $H^{p,p}(X,\mathbb{R})$ are equal to d_p.

Assume that properties 2 and 3 hold. Then, by Proposition 4.2.6, the currents

$$S_N^+ := \widehat{L}_N^+(\omega^p) \quad \text{where} \quad L_N^+ := \frac{1}{N}\sum_{n=1}^N d_p^{-n}(f^n)^*,$$

converge SP-uniformly to a positive closed (p,p)-current T^+. By Lemma 4.1.2, the cohomology class of T^+ is non-zero. Therefore, T^+ is a Green (p,p)-current. In the same way, we prove that

$$S_N^- := \widehat{L}_N^-(\omega^{k-p}) \quad \text{where} \quad \widehat{L}_N^- := \frac{1}{N}\sum_{n=1}^N d_p^{-n}(f^n)_*,$$

converge SP-uniformly to a Green $(k-p,k-p)$-current T^- of f^{-1}.

Since the convergences are SP-uniform, $S_N^+\wedge S_N^-$ converge to $T^+\wedge T^-$. Observe that $S_N^+\wedge S_N^-$ is a smooth positive measure. In order to prove property 1, we only have to check that the mass of $S_N^+\wedge S_N^-$ does not tend to 0. We have

$$\begin{aligned}\|S_N^+\wedge S_N^-\| &= \frac{1}{N^2}\sum_{1\leq n,l\leq N} d_p^{-n-l}\int_X (f^n)^*(\omega^p)\wedge (f^l)_*(\omega^{k-p})\\ &= \frac{1}{N^2}\sum_{1\leq n,l\leq N} d_p^{-n-l}\int_X (f^{n+l})^*(\omega^p)\wedge \omega^{k-p}\\ &= \frac{1}{N^2}\sum_{1\leq n,l\leq N} d_p^{-n-l}\|(f^{n+l})^*(\omega^p)\|.\end{aligned}$$

By Lemma 4.1.2, the last quantity is bounded from below by a positive constant independent of N. This implies that the mass of $S_N^+\wedge S_N^-$ does not tend to 0.

Now assume property 1 and let m denote the multiplicity of the spectral radius of f^* on $H^{p,p}(X,\mathbb{R})$. By Proposition 4.2.4, there are smooth closed (p,p)-form S^+ and $(k-p,k-p)$-form S^-, not necessarily positive, and an increasing sequence (n_i) such that

$$T^+ = \lim_{i\to\infty} n_i^{1-m}d_p^{-n_i}(f^{n_i})^*(S^+) \quad \text{and} \quad T^- = \lim_{i\to\infty} n_i^{1-m}d_p^{-n_i}(f^{n_i})_*(S^-).$$

Moreover, the convergences are SP-uniform. We have

$$\begin{aligned}\|T^+\wedge T^-\| &= \lim_{i\to\infty}\int_X n_i^{1-m}d_p^{-n_i}(f^{n_i})^*(S^+)\wedge n_i^{1-m}d_p^{-n_i}(f^{n_i})_*(S^-)\\ &= \lim_{i\to\infty}\int_X n_i^{2-2m}d_p^{-2n_i}(f^{2n_i})^*(S^+)\wedge S^-\\ &= \lim_{i\to\infty}\Big(\frac{2}{n_i}\Big)^{m-1}\int_X (2n_i)^{1-m}d_p^{-2n_i}(f^{2n_i})^*(S^+)\wedge S^-.\end{aligned}$$

The last integral can be computed cohomologically. By Lemma 4.1.2, it converges to a constant when i tends to infinity. Therefore, if $\|T^+\wedge T^-\|$ is strictly positive, $(2/n_i)^{m-1}$ does not converge to 0. It follows that $m=1$. □

When $p = 1$, using Hodge-Riemann theorem one prove easily that the properties in Proposition 4.4.1 are always satisfied. We don't know if this is the case in general. For this question, the reader will find some useful results and techniques developed in [33, 15, 39]. Here is the main result of this section. The property that μ is of maximal entropy was obtained in collaboration with de Thélin.

Theorem 4.4.2. *Let f be a holomorphic automorphism on a compact Kähler manifold (X, ω) of dimension k and d_s the dynamical degrees of f. Assume that there is a dynamical degree d_p strictly larger than the other ones and that f satisfies the properties in Proposition 4.4.1. Then f admits an invariant probability measure μ with Hölder continuous super-potentials. The measure μ is ergodic, hyperbolic and of maximal entropy. If the dominant eigenvalues of f^* on $H^{p,p}(X, \mathbb{C})$ are equal to d_p then μ is mixing.*

We will see that the last assertion holds under a weaker hypothesis: *d_p is the unique dominant eigenvalue which is a root of a real number.* This condition is always satisfied for some iterates of f. The measure μ is called *equilibrium measure* of f. By Propositions 3.4.4 and 3.4.5, μ is moderate and has positive Hausdorff dimension when X is projective. Since μ is hyperbolic, a theorem by Katok [38, p.694] says that any point in the support of μ can be approximated by saddle periodic points. Therefore, the saddle periodic points are Zariski dense in X since moderate measures have no mass on proper analytic subsets of X.

Recall that a positive invariant measure μ is *mixing* if for any test functions ϕ, ψ (smooth, continuous, bounded or in $L^2(\mu)$) we have

$$\langle \mu, (\phi \circ f^n)\psi \rangle \to \|\mu\|^{-1} \langle \mu, \phi \rangle \langle \mu, \psi \rangle.$$

The invariance of μ implies that $\langle \mu, (\phi \circ f^n)\psi \rangle = \langle \mu, \phi(\psi \circ f^{-n}) \rangle$. So, μ is mixing for f if and only if it is mixing for f^{-1}. Mixing is equivalent to the the property that $(\phi \circ f^n)\mu$ converge to a constant times μ. Indeed, since μ is invariant, we have $\langle (\phi \circ f^n)\mu, 1 \rangle = \langle \mu, \phi \rangle$; hence, the above constant should be $\|\mu\|^{-1} \langle \mu, \phi \rangle$. In fact, mixing is also equivalent to the property that any limit value of $(\phi \circ f^n)\mu$ is proportional to μ.

If μ is mixing then it is *ergodic*, that is, μ is extremal in the cone of positive invariant measures. This property is equivalent to the convergence

$$\frac{1}{N} \sum_{n=1}^{N} \langle \mu, (\phi \circ f^n)\psi \rangle \to \|\mu\|^{-1} \langle \mu, \phi \rangle \langle \mu, \psi \rangle$$

or to the property that any limit value of

$$\mu_N := \frac{1}{N} \sum_{n=1}^{N} (\phi \circ f^n)\mu$$

is proportional to μ. Note that μ is ergodic for f if and only if it is ergodic for f^{-1}. We refer to [38, 47] for the notions of entropy and of Lyapounov exponent. An invariant positive measure is *hyperbolic* if its Lyapounov exponents are non-zero.

We recall and introduce some notations that we will use. Let F and H be the dominant and strictly dominant subspaces of $H^{p,p}(X,\mathbb{R})$ for the action of f^*. Let $F^\vee$ and $H^\vee$ denote the dominant and strictly dominant subspaces of $H^{k-p,k-p}(X,\mathbb{R})$ for the action of f_*. By Theorem 4.2.1, we can associate to each class c in F or in H a current in $\mathcal{D}_p$ with Hölder continuous super-potentials that we denote by T_c^+. We can also apply this result to f^{-1} and associate to each class $c^\vee$ in $F^\vee$ or in $H^\vee$ a current $T_{c^\vee}^-$ in $\mathcal{D}_{k-p}$ with Hölder continuous super-potentials. By Proposition 3.4.2, the measure $T_c^+\wedge T_{c^\vee}^-$ has Hölder continuous super-potentials.

Denote by $\mathcal{M}$ the real space generated by $T_c^+\wedge T_{c^\vee}^-$ with $c\in F$ and $c^\vee\in F^\vee$ and $\mathcal{N}$ the real space generated by $T_c^+\wedge T_{c^\vee}^-$ with $c\in H$ and $c^\vee\in H^\vee$. We have

$$\dim\mathcal{M}\leq(\dim F)(\dim F^\vee)=(\dim F)^2$$

and

$$\dim\mathcal{N}\leq(\dim H)(\dim H^\vee)=(\dim H)^2.$$

Let $\mathcal{M}^+$, $\mathcal{N}^+$ be the closed convex cones of positive measures in $\mathcal{M}$ and in $\mathcal{N}$. The measure μ that we will construct is an extremal element of $\mathcal{N}^+$. We first prove the following lemmas.

Lemma 4.4.3. *For all $c\in F$ and $c^\vee\in F^\vee$, we have*

$$f^*(T_c^+\wedge T_{c^\vee}^-)=T_{f^*c}^+\wedge T_{f^*c^\vee}^-.$$

If c is in H and $c^\vee$ is in $H^\vee$, then $T_c^+\wedge T_{c^\vee}^-$ is an invariant measure.

Proof. Write as in Proposition 4.2.2

$$T_c^+=\lim_{i\to\infty}d_p^{-n_i}(f^{n_i})^*(S^+)\quad\text{and}\quad T_{c^\vee}^-=\lim_{i\to\infty}d_p^{-n_i}(f^{n_i})_*(S^-),$$

with S^+, S^- smooth. Observe that $d_p^{-n_i}(f^{n_i})^*[S^+]$ converge to the class c and $d_p^{-n_i}(f^{n_i})_*[S^-]$ converge to $c^\vee$. Hence, $d_p^{-n_i}(f^{n_i})^*[f^*(S^+)]$ converge to f^*c and $d_p^{-n_i}(f^{n_i})_*[f^*(S^-)]$ converge to $f^*c^\vee$. Applying Proposition 4.2.2 to $f^*(S^+)$ and to $f^*(S^-)$ yields

$$\begin{aligned}f^*(T_c^+\wedge T_{c^\vee}^-)&=f^*\big(\lim_{i\to\infty}d_p^{-n_i}(f^{n_i})^*(S^+)\wedge d_p^{-n_i}(f^{n_i})_*(S^-)\big)\\&=\lim_{i\to\infty}d_p^{-n_i}(f^{n_i})^*\big(f^*(S^+)\big)\wedge d_p^{-n_i}(f^{n_i})_*\big(f^*(S^-)\big)\\&=T_{f^*c}^+\wedge T_{f^*c^\vee}^-.\end{aligned}$$

When c is in H and $c^\vee$ is in $H^\vee$, we have $f^*c=d_pc$ and $f^*c^\vee=d_p^{-1}c^\vee$. Therefore, $T_{f^*c}^+=d_pT_c^+$ and $T_{f^*c^\vee}^-=d_p^{-1}T_{c^\vee}^-$. We deduce that $f^*(T_c^+\wedge T_{c^\vee}^-)=T_c^+\wedge T_{c^\vee}^-$. Since f is an automorphism, this also implies that $f_*(T_c^+\wedge T_{c^\vee}^-)=T_c^+\wedge T_{c^\vee}^-$. Hence, $T_c^+\wedge T_{c^\vee}^-$ is invariant. □

Lemma 4.4.4. *The cones $\mathcal{M}^+$ and $\mathcal{N}^+$ have non-empty interior in $\mathcal{M}$ and $\mathcal{N}$ respectively.*

Proof. Consider $c \in F$ and $c^\vee \in F^\vee$. Observe that if T_c^+ or $T_{c^\vee}^-$ is approximable by smooth positive closed currents then $T_c^+ \wedge T_{c^\vee}^-$ is positive and belongs to $\mathcal{M}^+$. We have seen that the sets of such classes have non-empty interiors in F and $F^\vee$. Hence, $\mathcal{M}$ is generated by such positive measures $T_c^+ \wedge T_{c^\vee}^-$. It follows that $\mathcal{M}^+$ has non-empty interior. The case of $\mathcal{N}^+$ is treated in the same way. □

Definition 4.4.5. Let μ be an invariant positive measure of f. We say that μ is *almost mixing* if there is a finite dimensional space V of measures such that for any function ϕ in $L^2(\mu)$ the limit values of $(\phi \circ f^n)\mu$, when $n \to \infty$, belong to V.

The above notion does not change if we use the continuous functions or the space $L^1(\mu)$ instead of $L^2(\mu)$ since continuous and L^2 functions are dense in $L^1(\mu)$. Note also that mixing corresponds to the case where V is of dimension 1. We will see in the following lemma that μ is almost mixing if and only if $L^2(\mu)$ can be decomposed into an invariant orthogonal sum $W \oplus W^\perp$ with $W^\perp$ of finite dimension such that $(\phi \circ f^n)\mu \to 0$ for $\phi \in W$. We can deduce that $(\psi \circ f^{-n})\mu \to 0$ for $\psi \in W$ and that μ is also almost mixing for f^{-1}. The following lemma is valid for a general dynamical system.

Lemma 4.4.6. *Let μ be a positive measure invariant by f. Assume that μ is almost mixing and that μ is ergodic for every f^n with $n \geq 1$. Then μ is mixing.*

Proof. Let V be the smallest space of measures such that for any real-valued continuous function ϕ the limit values of $(\phi \circ f^n)\mu$, when $n \to \infty$, belong to V. This space is invariant by f^* and f_*. We have to prove that $\dim V = 1$. Let W denote the space of functions $\psi \in L^2(\mu)$ with complex values such that $\langle \mu', \psi \rangle = 0$ for every $\mu' \in V$. Let $W^\perp$ denote the orthogonal of W. The spaces W, $W^\perp$ are invariant under f^*, f_* and we have $\dim_{\mathbb{C}} W^\perp = \dim V$. Moreover, continuous functions are dense in W. We show that $\dim_{\mathbb{C}} W^\perp = 1$.

Since f^* and f_* preserve the scalar product in $L^2(\mu)$, all the eigenvalues of f^* and of f_* have modulus equal to 1. So, if ψ is an eigenvector of f^* associated to an eigenvalue λ, we have $|\psi| \circ f = |\psi|$ and then $|\psi|$ is constant since μ is ergodic. Therefore, $\psi^n \in L^2(\mu)$ and $\psi^n \circ f = \lambda^n \psi^n$ for every $n \in \mathbb{Z}$. We claim that W does not contain any eigenvector. Otherwise, there is a function $\psi \in W \setminus \{0\}$ such that $\psi \circ f = \lambda \psi$ with $|\lambda| = 1$. Since ψ can be approximated by continuous functions in W, we deduce from the definition of W that for every $\phi \in L^2(\mu)$:

$$|\langle \psi\mu, \phi \rangle| = |\langle (\psi \circ f^{-n})\mu, \phi \rangle| = |\langle (\phi \circ f^n)\mu, \psi \rangle| \to 0.$$

We get that $\psi\mu = 0$, hence $\psi = 0$. This is a contradiction.

Consider now an eigenvector ψ of f^* in $W^\perp$ associated to an eigenvalue λ. Then, ψ^n is an eigenvector associated to λ^n for every $n \in \mathbb{Z}$. We deduce that ψ^n

is a function in $W^\perp$. Since $W^\perp$ is finite dimensional, λ is a root of unity. We have $\psi \circ f^n = \psi$ for some $n \geq 1$. Since μ is ergodic for f^n, ψ is constant. Hence, $\lambda = 1$ and it follows that $\dim_{\mathbb{C}} W^\perp = 1$ because f^* is an isometry of $W^\perp$. □

Consider the automorphism $\widetilde{f}$ of $X \times X$ given by $\widetilde{f}(x,y) := (f(x), f^{-1}(y))$. By Künneth formula, we have

$$H^{l,l}(X \times X, \mathbb{C}) \simeq \sum_{r+s=l} H^{r,s}(X,\mathbb{C}) \otimes H^{s,r}(X,\mathbb{C}).$$

Moreover, $\widetilde{f}^* \simeq (f^*, f_*)$ preserves this decomposition. It is shown in [14] that the spectral radius of f^* on $H^{r,s}(X,\mathbb{C})$ is bounded by $\sqrt{d_r d_s}$. We deduce that $d_k(\widetilde{f})$ is the maximal dynamical degree of $\widetilde{f}$. It is equal to d_p^2, with multiplicity 1 and is strictly larger than the other ones. So, the results obtained for f can be applied to $\widetilde{f}$. We will deduce several properties for f.

We use analogous notations $\widetilde{\mathcal{N}}$, $\widetilde{\mathcal{N}}^+$... for $\widetilde{f}$ instead of the notations $\mathcal{N}$, $\mathcal{N}^+$... for f. By Theorem 4.2.1 and Corollary 4.3.2 applied to $\widetilde{f}$, together with the Künneth formula, the family of the Green (k,k)-currents of $\widetilde{f}$ is a convex cone with non-empty interior in the real space generated by the currents $T_c^+ \otimes T_{c^\vee}^-$. The Green (k,k)-currents of $\widetilde{f}^{-1}$ is a convex cone with non-empty interior in the real space generated by the currents $T_{c^\vee}^- \otimes T_c^+$. Therefore, $\widetilde{\mathcal{N}}$ is generated by $\mu \otimes \mu'$ with μ, μ' in $\mathcal{N}$ and $\widetilde{\mathcal{M}}$ is generated by $\mu \otimes \mu'$ with μ, μ' in $\mathcal{M}$.

Let S^+ be a smooth current in $\mathcal{D}_p$ such that $d_p^{-n_i}(f^{n_i})^* S^+$ converge SP-uniformly to T_c^+ for some increasing sequence (n_i). Let S^- be a smooth current in $\mathcal{D}_{k-p}$ such that $d_p^{-n_i}(f^{n_i})_* S^-$ converge SP-uniformly to $T_{c^\vee}^-$. Then, we deduce from Proposition 4.2.2 applied to $\widetilde{f}$ that $d_p^{-2n_i}(\widetilde{f}^{n_i})^*(S^+ \otimes S^-)$ converge SP-uniformly to $T_c^+ \otimes T_{c^\vee}^-$. We will use this property in the computations involving $T_c^+ \otimes T_{c^\vee}^-$.

Lemma 4.4.7. *Let ϕ be a continuous real-valued function on X. If μ is a measure in $\mathcal{M}$, then any limit value of $(\phi \circ f^n)\mu$ is a measure in $\mathcal{M}$. In particular, the measures in $\mathcal{N}^+$ are almost mixing. If μ is in $\mathcal{N}$, then any limit value of*

$$\mu_N := \frac{1}{N} \sum_{n=1}^{N} (\phi \circ f^n)\mu$$

is a measure in $\mathcal{N}$.

Proof. Since continuous functions are uniformly approximable by smooth functions, we can assume that ϕ is smooth. We prove the first assertion. By definition of $\mathcal{M}$, we can assume that $\mu = T^+ \wedge T^-$ where T^+ is a (p,p)-current associated to a class c in F and T^- is a $(k-p,k-p)$-current associated to a class $c^\vee$ in $F^\vee$ as above. It is enough to show that if a subsequence $(\phi \circ f^{2n_i})\mu$ converge, then the limit is a measure in $\mathcal{M}$. Indeed, we obtain the case with odd powers by replacing ϕ by $\phi\circ f$.

Let ψ be another test smooth function on X. Define $\Phi(x,y):=\phi(x)\psi(y)$. Since μ is invariant, lifting the integrals on X to Δ we get

$$\begin{aligned}\langle(\phi\circ f^{2n})\mu,\psi\rangle &= \langle\mu,(\phi\circ f^n)(\psi\circ f^{-n})\rangle\\ &= \langle T^+\wedge T^-,(\phi\circ f^n)(\psi\circ f^{-n})\rangle\\ &= \langle(T^+\otimes T^-)\wedge[\Delta],\Phi\circ\widetilde{f}^n\rangle,\end{aligned}$$

where in order to obtain the last line we use a SP-uniform approximation of $T^+\otimes T^-$ by smooth currents as above. We have

$$\begin{aligned}\langle(\phi\circ f^{2n})\mu,\psi\rangle &= \langle(\widetilde{f}^n)_*(T^+\otimes T^-)\wedge(\widetilde{f}^n)_*[\Delta],\Phi\rangle\\ &= \big\langle(d_p^n(f^n)_*T^+\otimes d_p^n(f^n)^*T^-)\wedge d_p^{-2n}(\widetilde{f}^n)_*[\Delta],\Phi\big\rangle.\end{aligned}$$

Observe that $d_p^n(f^n)_*T^+$ belongs to a bounded family of currents constructed in Theorem 4.2.1. An analogous property holds for $d_p^n(f^n)^*T^-$. Therefore, the limit values of

$$(d_p^n(f^n)_*T^+\otimes d_p^n(f^n)^*T^-)\wedge d_p^{-2n}(\widetilde{f}^n)_*[\Delta]$$

are measures in $\widetilde{\mathcal{M}}$. It follows that $\langle(\phi\circ f^{2n_i})\mu,\psi\rangle$ converge to a finite combination of

$$\langle\mu^+\otimes\mu^-,\Phi\rangle=\mathrm{const}\langle\mu^-,\psi\rangle$$

with μ^+, μ^- in $\mathcal{M}$. We deduce that $(\phi\circ f^{2n_i})\mu$ converge to a combination of μ^-. So, the limit values of $(\phi\circ f^{2n})\mu$ are in $\mathcal{M}$. This completes the proof of the first assertion.

For the last assertion, we follow the same approach with T^+ associated to a class in H and T^- associated to a class in $H^\vee$. In this case, T^+, T^- are invariant and any limit value of

$$(T^+\otimes T^-)\wedge\frac{1}{N}\sum_{n=1}^{N}d_p^{-2n}(\widetilde{f}^n)_*[\Delta]$$

is a measure in $\widetilde{\mathcal{N}}$. We deduce as above that the limit values of μ_N are in $\mathcal{N}$. □

Proposition 4.4.8. *Let μ be a probability measure in $\mathcal{N}^+$. Then μ is ergodic if and only if it is an extremal element of $\mathcal{N}^+$. Moreover, the number of extremal probability measures in $\mathcal{N}^+$ is equal to $\dim\mathcal{N}$ and the convex cone $\mathcal{N}^+$ is generated by these measures. When d_p is the only dominant eigenvalue of f^* on $H^{p,p}(X,\mathbb{C})$ which is a root of a real number, then μ is mixing if and only if it is an extremal element of $\mathcal{N}^+$.*

Proof. If μ ergodic, μ is extremal in the cone of invariant positive measures. Therefore, μ is extremal in $\mathcal{N}^+$. Assume now that μ is extremal in $\mathcal{N}^+$. We show that it is ergodic. Let ϕ be a positive continuous function. The measures μ_N, defined as above, are positive and bounded by $\|\phi\|_\infty\mu$. By Lemma 4.4.7, any limit value of μ_N is a measure in $\mathcal{N}^+$ and it is bounded by $\|\phi\|_\infty\mu$. Since μ is extremal in $\mathcal{N}^+$, these limit values are proportional to μ. Therefore, μ is ergodic.

Recall that $\mathcal{N}^+$ is a salient convex closed cone in $\mathcal{N}$ with non-empty interior. Moreover, any element ν of $\mathcal{N}^+$ is an integral over extremal elements of mass 1. So, we get a decomposition of ν into ergodic probability measures. Since this decomposition is unique [47] and since $\mathcal{N}$ is generated by $\dim\mathcal{N}$ elements, we deduce that the number of extremal probability measures in $\mathcal{N}^+$ is equal to $\dim\mathcal{N}$ and the convex cone $\mathcal{N}^+$ is generated by these measures. So, $\mathcal{N}^+$ is a cone with simplicial basis.

Assume that d_p is the only dominant eigenvalue of f^* on $H^{p,p}(X,\mathbb{C})$ which is a root of a real number. Then the spaces H, $H^\vee$ do not change if we replace f by f^n. Therefore, $\mathcal{N}$ do not change if we replace f by f^n. We deduce that μ is ergodic for f^n. Lemmas 4.4.7 and 4.4.6 imply that μ is mixing. This completes the proof of the proposition. □

End of the proof of Theorem 4.4.2. Let μ be a probability measure which is an extremal element of $\mathcal{N}^+$. By Proposition 4.4.8, μ is ergodic and is mixing if d_p is the only dominant eigenvalue of f^* on $H^{p,p}(X,\mathbb{C})$ which is a root of a real number. By definition of $\mathcal{N}$, this measure has Hölder continuous super-potentials. It remains to prove that μ is of maximal entropy. Indeed, by a recent result of de Thélin [13], the property that μ is of entropy $\log d_p$ together with the fact that d_p is strictly larger than the other dynamical degrees implies that μ is hyperbolic, see also [16]. More precisely, μ admits p positive Lyapounov exponents larger than or equal to $\frac{1}{2}\log(d_p/d_{p-1})$ and $k-p$ negative exponents at most equal to $-\frac{1}{2}\log(d_p/d_{p+1})$.

The variational principle [47] implies that the entropy of an invariant measure is bounded from above by the topological entropy of f. By Gromov and Yomdin results [34, 48], the topological entropy of f is equal to $\log d_p$. Therefore, if ν is a probability measure in $\mathcal{N}^+$ then the entropy $h(\nu)$ of ν is at most equal to $\log d_p$. We will prove that $h(\nu) = \log d_p$ for every probability measure ν in $\mathcal{N}^+$.

Let S^+ be a smooth form in $\mathcal{D}_p$ and S^- a smooth form in $\mathcal{D}_{k-p}$. If S^+ and S^- are strictly positive, by Proposition 4.5.2 in the appendix below, any limit value ν of

$$\nu_n := \frac{1}{n}\sum_{l=1}^{n} d_p^{-n}(f^l)^*(S^+)\wedge(f^{n-l})_*(S^-)$$

is proportional to an invariant probability measure of maximal entropy $\log d_p$. By Proposition 4.2.2, the space M generated by these measures ν is of finite dimension. Let M_P denote the convex of probability measures in M. Since the entropy $h(\nu)$ is an affine function on ν [47, p.183], all the measures in M_P are of entropy $\log d_p$. It suffices to show that M contains $\mathcal{N}$. Observe that since ν_n depends linearly on S^+, S^-, the space M contains also the limit values of ν_n when S^+, S^- are not necessarily positive. When S^+ is in a class $c \in H$ and S^- is in a class $c^\vee \in H^\vee$, by Proposition 4.2.2, ν_n converge to $T^+_c \wedge T^-_{c^\vee}$. We deduce that M contains $\mathcal{N}$ and this implies the result. □

Remark 4.4.9. The property that the equilibrium measures are of maximal entropy can be proved using $\widetilde{f}$. More precisely, using Proposition 4.5.3 below for $Y := \Delta$, we can construct equilibrium measures of $\widetilde{f}$ with maximal entropy $2\log d_p$. This together with the Brin-Katok formula applied to f, f^{-1} and $\widetilde{f}$, see the appendix below and [47, p.99], implies that the equilibrium measures of f are of entropy $\log d_p$. The use of $\widetilde{f}$ may be a good method in order to study the distribution of periodic points of f by considering the intersection $(\widetilde{f^n})^*[\Delta] \wedge [\Delta]$.

Remark 4.4.10. The Green currents and the equilibrium measures have been constructed and studied by the authors in [20], for f with a dynamical degree d_p strictly larger than the other ones. Guedj considered in [35] the situation with the aditional hypothesis that d_p is the unique dominant eigenvalue of f^* on $H^{p,p}(X, \mathbb{C})$, i.e. $\dim F = \dim H = 1$. He claims that when X is projective, the equilibrium measure is of maximal entropy but he didn't give the proof. In this situation, we can find a subvariety Y of dimension p in X such that $d_p^{-n}(f^n)_*[Y]$ converge to a Green current T^-, see also [21]. Then, using the SP-uniform convergence $d_p^{-n}(f^n)^*(\omega^p) \to T^+$ or properties proved in [20], we deduce that $d_p^{-n-l}(f^n)^*(\omega^p) \wedge (f^l)_*[Y]$ converge to a constant times the equilibrium measure which, by Proposition 4.5.3 below, is of maximal entropy.

4.5 Appendix: measures of maximal entropy

This section contains an abstract construction of measures of maximal entropy. Most of the arguments given here are well-known, see Bedford-Smillie [2] and de Thélin [12]. For simplicity, assume that $f : X \to X$ is an automorphism as above which satisfies the properties in Proposition 4.4.1. The last hypothesis garantees that the construction gives non-zero measures. The method is still valid in a much more general setting, in particular, when f is a non-invertible finite map.

Given $\epsilon > 0$ and $n \in \mathbb{N}$, define *the Bowen ball* $B_n(a, \epsilon)$ by

$$B_n(a, \epsilon) := \big\{x \in X, \quad \mathrm{dist}(f^i(x), f^i(a)) \leq \epsilon \quad \text{for} \quad 0 \leq i \leq n\big\}.$$

Let ν be a probability measure invariant by f. By Brin-Katok [6], the function

$$h(\nu, a) := \sup_{\epsilon > 0} \lim_{n\to\infty} -\frac{1}{n} \log \nu(B_n(a, \epsilon))$$

is well-defined ν-almost everywhere and the entropy of ν is equal to

$$h(\nu) = \int h(\nu, a) d\nu(a).$$

We have the following Misiurewicz's lemma which is valid for continuous maps on compact metric spaces, see [2, 12, 47].

Lemma 4.5.1. *Let (n_i) be an increasing sequence of integers and ν_{n_i} probability measures such that the sequence*

$$\frac{1}{n_i}\sum_{l=0}^{n_i-1}(f^l)_*(\nu_{n_i})$$

converges to a measure ν. Assume there are constants $\epsilon > 0$ and $c_{n_i} > 0$ such that $\nu_{n_i}(B_{n_i}(a,\epsilon)) \leq c_{n_i}$ for all i and all Bowen ball $B_{n_i}(a,\epsilon)$. Then ν is an invariant probability measure and its entropy $h(\nu)$ satisfies the inequality

$$h(\nu) \geq \limsup_{i\to\infty} -\frac{1}{n_i}\log c_{n_i}.$$

We deduce from this lemma and an estimate due to Yomdin [48] the following proposition which was obtained in collaboration with de Thélin.

Proposition 4.5.2. *Let S^+ be a bounded positive (p,p)-form and S^- a bounded positive $(k-p,k-p)$-form on X, not necessarily closed. Assume there is an increasing sequence (n_i) of integers such that*

$$\frac{1}{n_i}\sum_{l=1}^{n_i} d_p^{-n_i}(f^l)^*(S^+)\wedge(f^{n_i-l})_*(S^-)$$

converge to a probability measure ν. Then ν is an invariant measure of maximal entropy $\log d_p$.

Proof. Denote by ν'_{n_i} the positive measure $d_p^{-n_i}(f^{n_i})^*(S^+)\wedge S^-$. Define $\nu_{n_i} := \lambda_{n_i}^{-1}\nu'_{n_i}$ where λ_{n_i} is the mass of ν'_{n_i}. Then ν_{n_i} are probability measures and we have

$$\lambda_{n_i}\frac{1}{n_i}\sum_{l=0}^{n_i-1}(f^l)_*(\nu_{n_i}) = \frac{1}{n_i}\sum_{l=1}^{n_i} d_p^{-n_i}(f^l)^*(S^+)\wedge(f^{n_i-l})_*(S^-)$$

which converge to the probability measure ν. We deduce that λ_{n_i} converge to 1. Therefore, by Lemma 4.5.1, it is enough to prove for any $0<\delta<1$ the existence of positive constants ϵ, A such that $\nu'_{n_i}(B_{n_i}(a,\epsilon)) \leq A d_p^{-n_i}e^{\delta n_i}$ for every $a\in X$. For this purpose, we can assume for simplicity that $S^+=\omega^p$ and $S^-=\omega^{k-p}$. We have to show that $\nu''_n(B_n(a,\epsilon)) \leq Ae^{n\delta}$ where $\nu''_n := (f^n)^*(\omega^p)\wedge\omega^{k-p}$. This inequality will be obtained by taking an average on an estimate due to Yomdin.

Let $Y\subset X$ be a complex manifold of dimension p smooth up to the boundary. If $\nu_n^Y := (f^n)^*(\omega^p)\wedge[Y]$ then $\nu_n^Y(B_n(a,\epsilon))$ is equal to the volume of $f^n(Y\cap B_n(a,\epsilon))$ counted with multiplicity. Yomdin proved in [48] that this volume is bounded by $Ae^{n\delta}$ when ϵ is small and A is large enough. The estimate is uniform on a and on Y. Now, consider a coordinate system $x = (x_1, \ldots, x_k)$ on a fixed chart of X with $|x_i| < 2$. In the unit polydisc D, up to a multiplicative constant, ω^{k-p} is bounded by $(dd^c\|x\|^2)^{k-p}$ which is equal to a combination of

$$(idz_{i_1} \wedge d\overline{z}_{i_1}) \wedge \ldots \wedge (idz_{i_{k-p}} \wedge d\overline{z}_{i_{k-p}}) \quad \text{with } 1 \leq i_1 < \cdots < i_{k-p} \leq k.$$

The last form is equal to an average on the currents of integration on the complex submanifolds of D which are given by

$$x_{i_1} = a_1, \quad \ldots, \quad x_{i_{k-p}} = a_{k-p} \quad \text{with } a_i \in \mathbb{C}.$$

So, by Yomdin's inequality, ν''_n restricted to D satisfies $\nu''_{n|D}(B_n(a, \epsilon)) \leq Ae^{n\delta}$ for some constants ϵ, A. Since X can be covered by a finite family of open sets D, we deduce that $\nu''_n(B_n(a, \epsilon)) \leq Ae^{n\delta}$ with $A > 0$. This completes the proof. □

One can prove in the same way the following proposition which is essentially due to Bedford-Smillie [2].

Proposition 4.5.3. *Let S be a continuous positive (p,p)-form, Y a complex manifold of dimension p in X smooth up to the boundary and χ a bounded positive function on Y. Assume there is an increasing sequence (n_i) such that*

$$\frac{1}{n_i} \sum_{l=1}^{n_i} d_p^{-n_i} (f^l)^*(S) \wedge (f^{n_i - l})_*(\chi[Y])$$

converge to a probability measure ν. Then ν is an invariant measure of maximal entropy $\log d_p$.

More general situations will be considered by de Thélin and Vigny in a forthcoming paper.

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