Chapter 1

Various Ways of Representing Surfaces and Basic Examples

Lecture 1.

a. First examples. For many people, one of the most basic images of a surface is the surface of the Earth. Although it looks flat to the naked eye (at least in the absence of any striking geographic features), we learn early in our lives that it is in fact round, and that its shape is very well approximated by a sphere. Geometrically, the sphere is defined as the locus of points at a fixed distance, called the *radius*, from a given point, the centre. Using Cartesian coordinates and putting the origin at the centre, we derive the familiar equation

$$x^2 + y^2 + z^2 = R^2, \tag{1.1}$$

where R is the radius; the sphere is the set of all points in $\mathbb{R}^3$ whose coordinates (x, y, z) satisfy this equation.

Many other familiar shapes can also be defined geometrically and represented as the set of solutions of a single equation, as in (1.1). For example, the (round) cylinder is the locus of points at a fixed distance from a given straight line. If the line is taken to be the z-axis and the

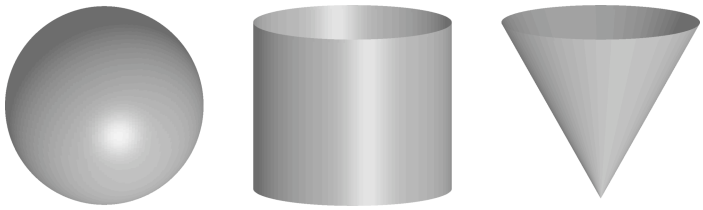


Figure 1.1. Three familiar surfaces.

distance is equal to R, the equation for the cylinder is

$$x^2 + y^2 = R^2. \tag{1.2}$$

Another surface familiar from elementary geometry (and also from ice-cream parlours) is the cone, which is obtained by rotating a straight line around another line which intersects it. If the axis of rotation is again the z-axis and the initial line lies in the xz-plane, with the equation $x = az$, then the cone is given by the equation

$$x^2 + y^2 = a^2 z^2. \tag{1.3}$$

Exercise 1.1. If we construct a surface of revolution using parallel lines instead of intersecting lines (as we did with the cone), we obtain a cylinder. There is a third possibility; the lines may be *skew*, that is, neither intersecting nor parallel. Describe the surface obtained in this case, and derive its equation.

We feel immediately that the three objects expressed by equations (1.1), (1.2), and (1.3), which are shown in Figure 1.1, are very different in a variety of robust ways. For example, the sphere is bounded—in fact, compact—while the cylinder and cone are not (contrary to what the picture might suggest). The sphere and cylinder are smooth everywhere, while the cone has a special point, the intersection of the two lines in the construction, which is the origin in (1.3).

These differences are qualitative, and would not be changed if we deformed each surface by a small amount—this reflects the fact that the three surfaces in question have different *topologies*. Such a deformation would, however, change the quantitative properties of a surface, which constitute its *geometry*. For example, stretching or

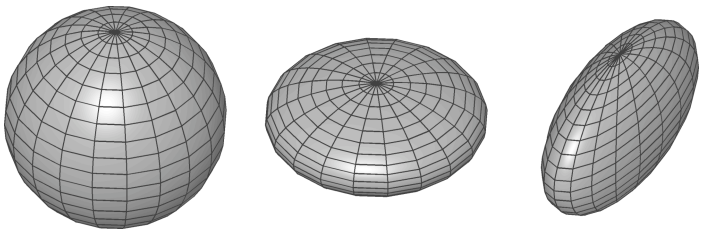


Figure 1.2. Three ellipsoids.

squeezing the sphere along the three coordinate axes produces an ellipsoid given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{1.4}$$

where a, b, and c are parameters which depend on the degree of stretching or squeezing. Of the three surfaces above, the overall shape and crude properties of an ellipsoid (its topology) are most similar to that of a sphere, and are quite different from that of a cylinder or a cone; its geometry, however, displays many differences from the geometry of a sphere.[1] For example, the sphere has many symmetries (that is, rigid motions of the space which leave the sphere as a whole in place), while a triaxial ellipsoid (one for which all three numbers a, b, and c in (1.4) are different, such as the third shape shown in Figure 1.2) has only a few.

Exercise 1.2. Find all the symmetries for

(1) a triaxial ellipsoid;

(2) an ellipsoid of revolution for which $a = b \neq c$ (such as the second ellipsoid in Figure 1.2).

Consider separately the symmetries which can be effected by a continuous motion of the space and those which cannot, such as reflections with respect to planes.

[1]For the time being, we rely on intuitive ideas of what constitutes a general shape. For a reader steeped in mathematical rigor, we refer to notions of homeomorphism and diffeomorphism, which will be introduced later in Lectures 4 and 17, respectively, and say that two surfaces have similar shapes if they are homeomorphic, or diffeomorphic in the case of smooth surfaces.

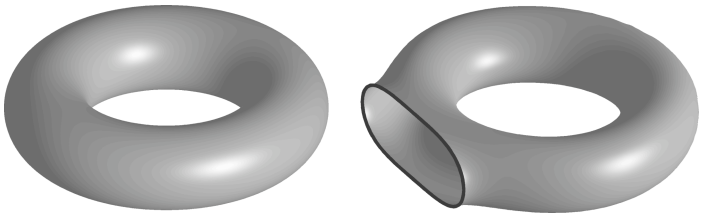


Figure 1.3. A torus and a handle.

Another familiar example of a surface is a torus—just as the sphere is the surface of a idealised ball, the torus is the surface of an idealised doughnut (or perhaps a bagel, depending on what sort of diet one is on). Like our first three examples, it is a surface of revolution, and may be obtained by rotating a circle around a line which lies in the plane of the circle, but does not intersect it. We will derive a nice equation (1.5) for the torus in the next lecture.

We can obtain new surfaces with qualitatively distinctive shapes by the procedure called "attaching a handle". A handle can be thought of as a torus with a hole (or if you like, an inner tube with a small patch cut out), as shown in Figure 1.3—this is attached to a hole cut in a given surface. Applying this procedure to a sphere produces a surface in the general shape of a torus. If we continue to attach more handles, we obtain something reminiscent of a pretzel with an increasing number of holes or, alternatively, a chain of tori linked to each other—Figure 1.4 shows a sphere with two handles. Like all the surfaces we have dealt with so far, these surfaces can also be represented by equations with a certain amount of effort (see Exercise 1.6).

b. Equations vs. other methods. We have obtained several different surfaces as the set of points whose coordinates (x, y, z) satisfy one equation or another. It is natural to ask what sort of equations will always yield nice, recognisable surfaces. Will any old equation do? Or must we impose some restrictions? And conversely, can we represent every surface by an equation?

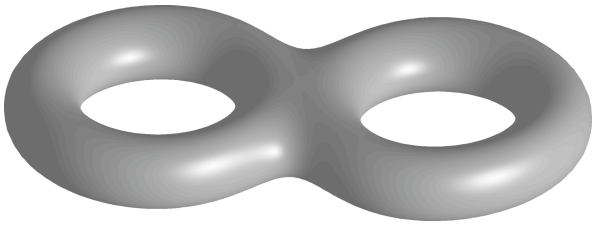


Figure 1.4. A sphere with two handles.

We begin by asking what sorts of equations are acceptable. By moving all the terms to the same side, any equation in x, y, and z can be written in the form $F(x, y, z) = 0$. If we hope to get a smooth surface, we must demand that the function F is at least differentiable—any of the equations (1.1), (1.2), (1.3), and (1.4) can be written in this form with a quadratic polynomial as the function F. But why are the sphere, the cylinder, and the ellipsoid all smooth, while the cone has a special point? The difference is clearly seen in the geometric description of the surfaces, since the line we use to define the cone passes through the axis of rotation, but it is not so easy to see what feature of the equations is responsible. How does this point of non-smoothness turn up in the equations?

The answer is that the origin is a *critical point* of the function $x^2+y^2-a^2z^2$ and lies on the surface defined by (1.3), while the other functions—$x^2 + y^2 + z^2 - R^2$, $x^2 + y^2 - R^2$, and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$—have no critical points at the zero level. Thus, if we want to define a smooth surface in $\mathbb{R}^3$ by an equation of the form $F(x, y, z) = 0$, the function F should have no critical points at the zero level.

Turning to the other half of the relationship between surfaces and equations, we find that not every geometric object which common sense would call a surface can be represented as the solution set of an equation. One difficulty is caused by boundaries—notice that the cylinder defined in (1.2) is unbounded, and extends infinitely far in both the positive and negative z-directions. Suppose we want to consider a finite cylinder, which may be obtained by rotating an interval around a parallel line, or by rolling up a rectangular sheet of paper

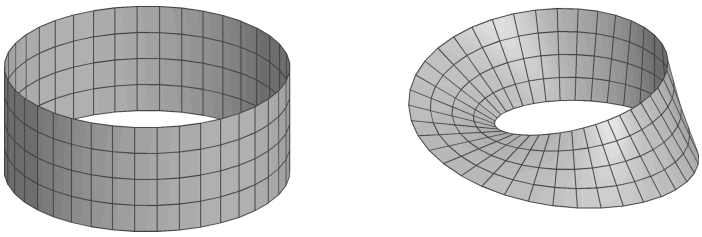


Figure 1.5. Two ways of gluing ends together.

and gluing together two opposite edges. How are we to represent such a surface by an equation?

One possibility is to add an auxiliary inequality—for example, one particular bounded cylinder is given as the solution set of

$$x^2 + y^2 = R^2, \qquad z^2 \leq 1$$

This method solves the problem in some cases, but not all. Consider the second description of a cylinder given above, in which we take a band of paper and glue together the two ends—now look at what happens if we twist the band halfway around before gluing the ends together! The result is the famous *Möbius band* (or *Möbius strip*), shown in Figure 1.5. Its most surprising property is that it only has one side: an insect which crawls once around the band will find itself at the same place, but on the opposite side of the surface.

Now any surface which is given by an equation $F(x, y, z) = 0$ (with or without inequalities) and which does not contain any critical points must have two sides—the function F is positive on one side and negative on the other. It follows that the Möbius strip cannot be represented as the solution set of a 'nice' equation in the sense discussed above.

A related counterintuitive property of the Möbius strip has to do with closed curves. In the plane, any closed curve divides the plane into two regions[2]—on the Möbius strip, though, we can draw closed curves which have no "inside" or "outside". Consider the curve which divides the strip in half, so to speak, running halfway between the free

[2]This if the *Jordan Curve Theorem*, which we will state and prove rigorously in Lectures 34 and 35. It is not as easy as one might first think!

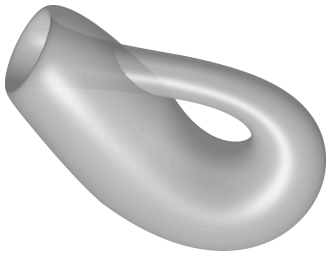


Figure 1.6. Immersing a Klein bottle in $\mathbb{R}^3$.

edges. If we take a pair of scissors and cut along this curve, we will be left with a single connected surface, rather than two disconnected pieces, which is what would happen if we performed the same operation on the cylinder, for example. This fact is intimately connected to the observation that if we place a clock at some point on this curve and move it once around the strip, when it returns it will be running counterclockwise!

The existence of the Möbius strip is the first indication that representing surfaces by equations is not sufficient. In the next lecture we will discuss an alternative way of representing it in an analytical fashion. Notice, however, that the Möbius strip, along with all our other examples, still lives comfortably in three-dimensional Euclidean space. Our next example challenges the assumption that all interesting surfaces can be realised this way.

If we want to glue together two opposite sides of a rectangle, we can either glue them with no twist, which produces a cylinder, or with a half-twist, which produces a Möbius strip.[3] A similar dichotomy arises if we decide to glue together the two ends of a cylinder. If we do this in the conventional way, we produce a torus—however, this is only one of two possible alignments for the pair of circles which are to be attached. The second possibility involves 'flipping' one of the ends around somehow, and results not in a torus, but in a *Klein bottle*. The closest we can come to visualising this in three dimensions is to have one end approach the other end not from outside the cylinder,

[3]A second half-twist will produce something which turns out to be homeomorphic to a cylinder, but with a different embedding in $\mathbb{R}^3$.

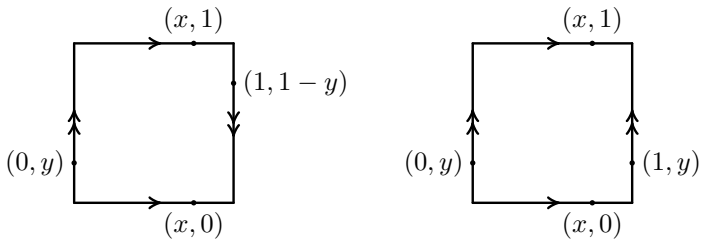


Figure 1.7. Planar models of a Klein bottle and a torus.

as with the torus, but from *inside*—to accomplish this, we must pass the end through the wall of the cylinder, creating a sort of twisted bottle (hence the name), as shown in Figure 1.6.

c. Planar models. Unlike the earlier examples, the Klein bottle cannot be embedded in $\mathbb{R}^3$, and so it is more difficult to represent properly. Abstractly, however, the procedure we followed to create it is not hard to describe, and this idea introduces a totally different way of looking at surfaces. We begin by taking the unit square for our rectangle:

$$X = \{\, (x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ 0 \le y \le 1 \,\}.$$

We may then 'glue' together two opposite edges by declaring that for each value of x between 0 and 1, the pair of points $(x,0)$ and $(x,1)$ are now the same point. This gives an abstract representation of the cylinder—to obtain a Klein bottle, we must 'glue' together the two remaining edges with a flip.[4] We do this by considering each pair of points $(0,y)$ and $(1,1-y)$ as a single point—notice that all four corners are now identified. One easily checks that a piece of this object near every point looks like a piece of ordinary plane, so this seems to be a legitimate surface.[5]

Now we can look at the procedure just described and contemplate what happens when we identify both pairs of sides of the square in the conventional way—$(x,0)$ with $(x,1)$ and $(0,y)$ with $(1,y)$. We

[4]These edges are now "circles", in the topological sense at least, since $(0,0)$ and $(0,1)$ are the same point, and similarly for $(1,0)$ and $(1,1)$.

[5]Of course, we have not defined rigorously what we mean by a 'legitimate surface'. A two-dimensional smooth manifold (see Lecture 16) certainly qualifies.

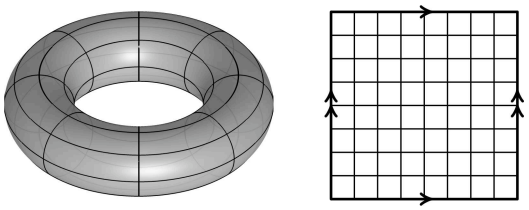


Figure 1.8. Meridians and parallels on two tori with different geometries.

obtain a surface resembling a torus as far as its global properties are concerned. For example, vertical and horizontal segments become closed curves which are identified with "parallels" and "meridians" of the torus of revolution—this will become clear in the next lecture when we introduce parametric representations of surfaces. However, the geometry of our surface, the *flat torus*, is different from that of the torus of revolution. For example, all vertical and all horizontal "circles" in the flat torus have the same length, while in the torus of revolution the meridians have the same length but the parallels do not (Figure 1.8). This is a consequence of the fact that although the cylinder in $\mathbb{R}^3$ has the same intrinsic geometry as the sheet of paper with only one pair of sides identified (that is, the paper is not stretched), it cannot be bent in $\mathbb{R}^3$ without a distortion. So far, our notion of internal geometry is intuitive, but soon we will make it more precise.

Let us try to exhaust the possibilities of surface-building from a rectangular piece of paper. The only remaining way of identifying pairs of opposite sides is to identify both pairs of sides using a flip, so that we identify $(x, 0)$ with $(1 - x, 1)$ and $(0, y)$ with $(1, 1 - y)$. We will now turn our attention to this construction.

Exercise 1.3. Describe the surface obtained from the square by identifying points on pairs of adjacent sides, i.e. $(0, t)$ with $(1 - t, 1)$ and $(1, t)$ with $(1-t, 0)$. Pay attention both to the shape and to geometry.

d. Projective plane and flat torus as factor spaces. To get a more symmetric picture for the last construction, we may inflate the square to a disc into which the square is inscribed, project the boundary of the square radially to the circumference of the disc, and observe

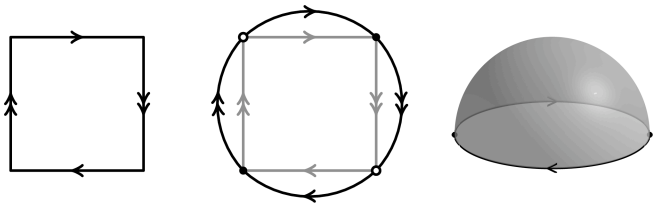


Figure 1.9. Various models for the real projective plane.

that the identified pairs become antipodal points on the boundary circle. Thus our object becomes the disc with pairs of opposite points on the boundary identified, as in Figure 1.9. To make this even more symmetric, inflate the disc to a hemisphere, keeping the boundary as the equator. Now we can add the other hemisphere and observe that each point of our object is represented by a pair of opposite points on the sphere.

Instead of taking pairs of antipodal points as the points of our surface, we may observe that any such pair determines a unique line in $\mathbb{R}^3$ passing through the centre of the sphere, and vice versa. Thus we may also think of our surface as the set of all lines through a particular point—the surface so obtained is called the *projective plane*, denoted $\mathbb{R}P^2$. An obvious advantage of the sphere representation over gluing is that it highlights the uniformity of the surface; all points look the same.

Inspired by the last construction, we may try to look at the flat torus differently. First recall that the circle can be represented either by an interval, say $[0, 1]$, with endpoints identified, or as the set of equivalence classes of real numbers modulo one, i.e. the set of all fractional parts of real numbers. If we simply think of all numbers with the same fractional part as the same element of the circle we come to the representation $S^1 = \mathbb{R}/\mathbb{Z}$—note that here every point on the circle is represented in the same way, in contrast to the interval with endpoints identified, where the choice of representation led to a false distinction between endpoints and non-endpoints. This choice of representation is a matter of fixing a *fundamental domain*; that is, a subset of $\mathbb{R}$ which contains exactly one element of each equivalence

class, except along its boundary, where it may contain two or more. In this case, we may take any unit interval as our fundamental domain.

A similar observation may be made with two variables, where we observe that the (flat) torus $\mathbb{T}^2$ can be identified with the set of pairs of fractional parts of real numbers:

$$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2,$$

where $\mathbb{Z}^2$ is the lattice of vectors with integer coordinates. These equivalence classes are represented by points in the unit square (the fundamental domain), once pairs of boundary points whose difference is an integer have been identified.

We may make one further step into abstraction; instead of vectors with integer coordinates, think about translations by those vectors. Then each equivalence class in $\mathbb{R}^2/\mathbb{Z}^2$ becomes an orbit of the group of such translations acting on $\mathbb{R}^2$, and our factor space (or *quotient space*) naturally becomes the space of orbits.

The same approach may be taken with the projective plane—notice that the flip on the sphere is a transformation which generates a group of two elements, since its square is the identity. The orbit of a point under the action of this group consists of the point itself, together with its antipode—identifying each such pair of points yields the projective plane, which can thus be thought of as the space of orbits of this two-element group acting on the sphere.

Exercise 1.4. Represent the cylinder, the infinite Möbius strip, and the Klein bottle as orbit spaces for some groups acting on the Euclidean plane $\mathbb{R}^2$. The infinite Möbius strip is the infinite rectangle $[0,1] \times \mathbb{R}$ with each pair of points $(0, y)$ and $(1, -y)$ identified.

Lecture 2.

a. Equations for surfaces and local coordinates. Consider the problem of writing an equation for the torus; that is, finding a function $F\colon \mathbb{R}^3 \to \mathbb{R}$ such that the torus is the solution set $\{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0\}$. Because the torus is a surface of revolution, we begin with the equation for a circle in the xz-plane with radius 1 and centre at $(2, 0)$:

$$S^1 = \left\{ (x, z) \in \mathbb{R}^2 \;\middle|\; (x-2)^2 + z^2 = 1 \right\}$$

To obtain the surface of revolution, we replace x with the distance from the z-axis by making the substitution $x \mapsto \sqrt{x^2+y^2}$, and obtain

$$\mathbb{T}^2 = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; (\sqrt{x^2+y^2}-2)^2 + z^2 - 1 = 0 \right\}$$

At first glance, then, setting $F(x,y,z) = (\sqrt{x^2+y^2}-2)^2 + z^2 - 1$ gives our desired solution. However, this suffers from the defect that F is not differentiable along the z-axis; we can overcome this fairly easily with a little algebra. Expanding the equation, isolating the square root, and squaring both sides, we obtain

$$\begin{aligned} x^2+y^2+4-4\sqrt{x^2+y^2}+z^2-1 &= 0 \\ x^2+y^2+z^2+3 &= 4\sqrt{x^2+y^2} \\ (x^2+y^2+z^2+3)^2 &= 16(x^2+y^2) \end{aligned}$$

and hence consider the function F defined by

$$F(x,y,z) = (x^2+y^2+z^2+3)^2 - 16(x^2+y^2). \tag{1.5}$$

It is easy to check that the new choice of F from (1.5) does not introduce any extraneous points to the solution set, and now F is differentiable on all of $\mathbb{R}^3$.

Exercise 1.5. Prove that a sphere with $m \geq 2$ handles cannot be represented as a surface of revolution.

Due to the result in Exercise 1.5, this argument cannot be applied directly to find an equation whose set of solutions look like a sphere with $m \geq 2$ handles, but we can reverse engineer the result to find a general method. Instead of beginning with a vertical plane, we consider the intersection of the torus and the horizontal xy-plane, which is given by two concentric circles. $F(x,y,0)$ is negative between the circles, hence $F(x,y,z) = F(x,y,0) + z^2 = 0$ has two solutions for those values of x and y, leading to the torus shape. By beginning with three or more circles (no longer concentric) we may use this idea to represent a sphere with any number of handles.

Exercise 1.6. Represent a sphere with two handles as the set of solutions of the equation $F(x,y,z) = 0$, where F is a differentiable function, and none of its critical points satisfy this equation.

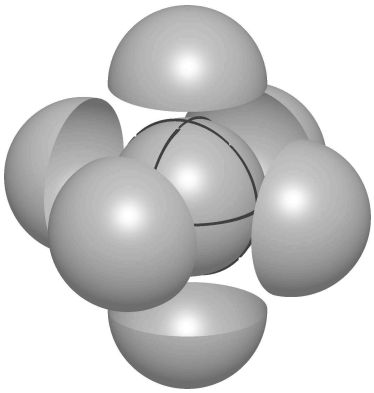


Figure 1.10. The sphere as a union of graphs.

What good is all this? What benefit do we gain from representing the torus, or any other surface, by an equation? Of course, it allows us to plug the equation into a computer and look at pretty pictures of our surface, but what we are really after is *coordinates* on our surface. After all, the surface is a two-dimensional affair, and so we should be able to describe its points using just two variables, but the equations we obtain are written in three variables.

To address this, we first backtrack a bit and discuss graphs of functions. Recall that given a function $f\colon \mathbb{R}^2 \to \mathbb{R}$, the graph of f is

$$\text{graph } f = \{\, (x, y, z) \in \mathbb{R}^3 \mid z = f(x, y) \,\}$$

If f is 'nice', its graph is a 'nice' surface sitting in $\mathbb{R}^3$. Of course, most surfaces cannot be represented globally as the graph of such a function; the sphere, for instance, has two points on the z-axis, and hence we require at least two functions to describe it in this manner.

In fact, more than two functions are required if we adopt this approach. The unit sphere is given as the solution set of $x^2+y^2+z^2 = 1$, so we can write it as the union of the graphs of f_1 and f_2, where

$$f_1(x, y) = \sqrt{1 - x^2 - y^2}$$
$$f_2(x, y) = -\sqrt{1 - x^2 - y^2}$$

The graph of f_1 is the northern hemisphere, the graph of f_2 the southern. However, we run into problems at the equator $z = 0$; for reasons which will be made apparent when we give the precise definition of a manifold (topological or differentiable), it is important that the domain on which we define each graph be *open*. In this particular case, this means we cannot include the equator in either the northern or the southern hemisphere, and must cover those points with other graphs. By using graphs with x or y as the dependent variable, we can cover the 'eastern' and 'western' hemispheres, as it were, but find that we require six graphs to deal with the entire sphere, as shown in Figure 1.10.

This approach has wide validity. Recall that $(x, y, z) \in \mathbb{R}^3$ is a *critical point* of a smooth function $F\colon \mathbb{R}^3 \to \mathbb{R}$ if the gradient of F vanishes at (x, y, z), and that a point is called *regular* if it is not critical. If S is the zero set of such a function, then at any regular point in S we can apply the Implicit Function Theorem and obtain a neighbourhood of the point which is the graph of some function; in essence, we are projecting patches of our surface to the various coordinate planes in $\mathbb{R}^3$. If our surface contains only regular points, this allows us to describe the entire surface in terms of these local coordinates.

As indicated in the first lecture, if the gradient vanishes at a point, the set of solutions may not look like a nice surface. A trivial example is the sphere of radius zero, $x^2 + y^2 + z^2 = 0$; a more interesting example is the cone $x^2 + y^2 - z^2 = 0$ near the origin.

b. Other ways of introducing local coordinates. From the geometric point of view, the choice of planes involved in representing a surface as the union of graphs of functions is somewhat arbitrary and unnatural; for example, the orthogonal projection of the northern hemisphere of S^2 to the xy-plane represents points in the 'arctic' quite well, but distorts things rather badly near the equator, where the derivative of the function blows up. If we are interested in angles, distances, and other geometric qualities of the surface, a more natural choice is to project to the tangent plane at each point; this will lead us eventually to the notion of a *Riemannian manifold*. If the previous approach represented an effort to draw a 'world map' of

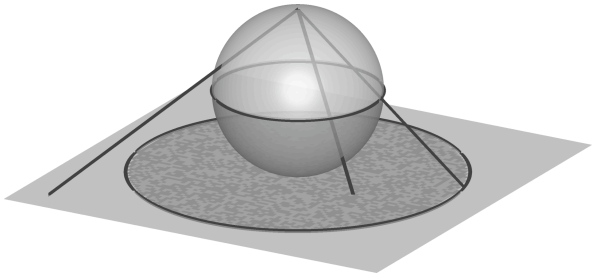


Figure 1.11. Stereographic projection from the sphere to the plane.

as much of the surface as possible, without regard to distortions near the edges, this approach represents publishing an atlas, with many smaller maps, each zoomed in on a small neighbourhood of each point in order to minimise distortions.

Orthogonal projections, whether to coordinate planes or tangent planes, form only a subset of the class of local coordinates on surfaces; there are many other members of this class besides. In the case of a sphere, one well-known example of local coordinates is stereographic projection (Figure 1.11), which gives a diffeomorphism[6] from the sphere minus a point to the plane.

Another example is given by the use of the familiar system of longitude and latitude to locate points on the surface of the earth; these resemble polar coordinates, mapping the sphere minus a point onto the open disc (Figure 1.12). The north pole is the centre of the disc, while the (deleted) south pole is its boundary; lines of longitude (meridians) become radii of the disc, while lines of latitude (parallels) become concentric circles around the origin.

However, if we want to measure distances on the sphere using any of these local coordinates, we cannot simply use the usual Euclidean distance in the disc or the plane—for example, the polar coordinates mentioned in the last example preserve distances along lines of longitude (radii), but distort distances along lines of latitude (circles centred at the origin). This is especially true near the boundary of the

[6]That is, a bijective differentiable map with differentiable inverse. See Lecture 17 for more details.

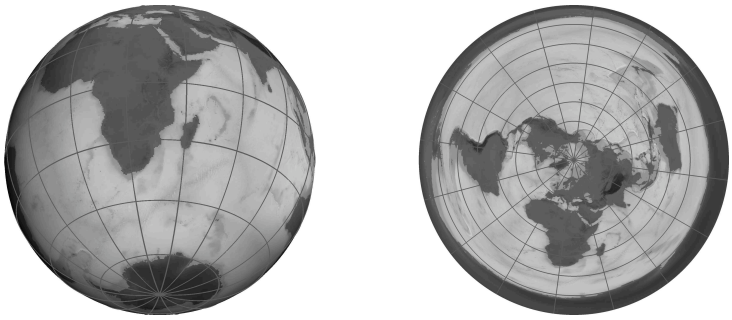


Figure 1.12. From the sphere to a disc via geographic coordinates.

disc, where the actual distance between points is much less than the Euclidean distance (since every point on the boundary is identified)—notice how much Antarctica is stretched out in Figure 1.12. This gives us our first example of a *Riemannian metric* (which for the time being we may simply think of as a notion of distance) on $\mathbb{D}^2$, apart from the usual Euclidean one.

Exercise 1.7. Stereographic projections from the north and south poles introduce two coordinate systems on the sphere minus the poles. Find the coordinate transformation from one of those systems to the other—that is, if a point on the sphere has coordinates (x, y) in the coordinate system projected from the north pole and (x', y') in the projection from the south, find (x', y') as a function of (x, y).

c. Parametric representations. While the idea of putting local coordinates on a surface will turn out to be more useful in general, we will occasionally have reason to deal with parametric representations. There are two important distinctions between these two methods of introducing coordinates on a surface.

First, local coordinates involve a map from the surface to a plane domain, while a parametric representation is a map from a plane domain to the surface. Formally, then, these two constructions are mutual inverses.

The second distinction is that a local coordinate system usually does not attempt to cover the entire surface by a single coordinate

system, but rather uses several patches to accomplish the task. A parametric representation, on the other hand, usually involves a map from a plane domain to a surface which is onto, or at at least nearly so, as in the inverse to the stereographic projection. One should also keep in mind that, while the notion of an atlas of local coordinate systems has a precise meaning which we will describe in Chapter 3, the notion of parametric representation is somewhat vague.

Exercise 1.8. Write a parametric representation of the torus of revolution (1.5) using the 'latitude' (position of a plane section) and 'longitude' (the angular coordinate along a plane section) as parameters. Use this representation to construct a bijection between the flat torus from Lecture 1(d) and the torus of revolution.

d. Metrics on surfaces. As our discussion of local coordinates suggested, we must address the question of how the distance between two points on a surface is to be measured. In the case of the Euclidean plane, we have a formula, obtained directly from the Pythagorean theorem. For points on the sphere of radius R we also have a formula: the distance between two points is simply the angle they make with the centre of the sphere, multiplied by R. Properties of this distance, such as the triangle inequality, can be deduced via elementary geometry, or by representing the points as vectors in $\mathbb{R}^3$ and using properties of the inner product.

These explicit formulae are serendipitous consequences of the extremely symmetric shapes of the plane and the sphere. What is the correct notion of distance on an arbitrary surface? Recalling that in the plane at least, the shortest path between two points is a straight line, and it is precisely along this line that the distance given by the Pythagorean theorem is measured, we may suggest that the distance between two points should naturally be defined as the length of the shortest path connecting them.

In general, since we do not yet know whether such a shortest path always exists, the proper definition of distance is as the infimum of the set of lengths of paths connecting the two points. Of course, this requires that we have a definition for the length of a path on the surface. We can find the length of a path in $\mathbb{R}^3$ by approximating it with piecewise linear paths and then using the notion of distance

in $\mathbb{R}^3$, which we already know. If our surface is not embedded in Euclidean space, however, we must replace this with an infinitesimal notion of distance, the Riemannian metric alluded to above. We will give a precise definition and discuss examples and properties of such metrics later in this course.

Lecture 3.

a. More about the Möbius strip and projective plane. Let us go back to the Möbius strip. The most common way of introducing it is as a sheet of paper (or belt, carpet, etc.) whose ends have been attached after giving one of them a half-twist. In order to represent this surface parametrically, it is useful to consider the factor space construction, which was discussed in the first lecture for the Klein bottle and the flat torus, and which is even simpler in the case of the Möbius strip.

Begin with a rectangle R. We are going to identify each point on the left-hand vertical boundary of R with a point on the right-hand boundary; if we identify each point with the point directly opposed to it (on the same horizontal line), we obtain a cylinder. To obtain the Möbius strip, we identify the lower left corner with the upper right corner and then move inwards; in this fashion, if $R = [0,1] \times [0,1]$, the point $(0,t)$ is identified with the point $(1,1-t)$ for $0 \leq t \leq 1$.

To embed this in $\mathbb{R}^3$, we can effect the half-twist by a continuous uniform rotation of an interval (the vertical lines in the model) whose centre moves around a closed curve (say a circle), and which remains perpendicular to that circle. Using the x-coordinate in the model as the angular coordinate along the circle, and the y-coordinate as the distance along the interval, one can write a parametric representation of a Möbius strip in $\mathbb{R}^3$ (see Figure 1.5).

Exercise 1.9. Write explicit expressions for the parametric representation of a Möbius strip embedded into $\mathbb{R}^3$ without self-intersections described above.

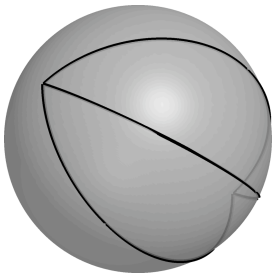


Figure 1.13. Multiple geodesics between antipodal points.

The projective plane with distance inherited from the sphere[7] is called the *elliptic plane*—it will be one of the star exhibits of this course. We can motivate its definition by considering the sphere as a geometric object, on which the notion of a line in Euclidean space is to be replaced by the concept of a *geodesic*; one key property of the former is that it is the shortest path between two points, and so informally at least, geodesics are simply curves which have this property. On the sphere, we will see that the geodesics are great circles, and so we may attempt to formulate various geometric propositions in this setting. However, this turns out to have some undesirable features from the point of view of conventional geometry; for example, every pair of geodesics intersects in *two* (diametrically opposite) points, not just one. Further, any two diametrically opposite points on the sphere can be joined by infinitely many geodesics (Figure 1.13), in stark contrast to the "two points determine a unique line" rule of Euclidean geometry.

Both of these difficulties are related to pairs of diametrically opposed points; the solution turns out to be to identify such points with each other. Identifying each point on the sphere with its antipode yields a quotient space, which is the projective plane described at the end of the first lecture. Alternatively, we can consider the flip map $I\colon (x, y, z) \mapsto (-x, -y, -z)$, which is an isometry of the sphere without fixed points. Declaring all members of a particular orbit of I to

[7]This simply means that the distance between two points in the projective plane is taken to be the minimum of pairwise distances between points in the sphere representing those points.

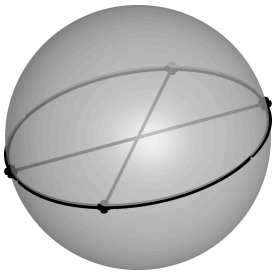


Figure 1.14. Determining distances in $\mathbb{R}P^2$ via central angles.

be the same point, we obtain the quotient space S^2/I, which is again the projective plane, or the elliptic plane when we are interested in the geometry.

In the elliptic plane, there is no such notion as the sign of an angle; we cannot consistently determine which angles are positive and which are negative. All the other geometric notions carry over, however; the distance between two points can still be found as the magnitude of the (acute) central angle they make (Figure 1.14), and the notions of angle between geodesics and length of geodesics are still well-defined.

Exercise 1.10. Write at least five propositions from Euclidean geometry which are true in the elliptic plane and at least three propositions which are true in Euclidean geometry and are not true in the elliptic plane. Each proposition must include statements about configurations of lines and/or isometries, and no two should be trivial reformulations of each other.

b. A first glance at geodesics. Informally, as mentioned above, a *geodesic* is the curve of shortest length between two points; more precisely, it is a curve γ with the property that given any two points $\gamma(a)$ and $\gamma(b)$ whose parameter values a and b are sufficiently close together, any other curve from one point to the other will have length at least as great as the portion of γ between the two. Later in the course (Lecture 25), we will consider the question of whether such a curve always exists between two points, and whether it is unique.

The two most basic examples are the Euclidean spaces $\mathbb{R}^n$, where geodesics are straight lines, and the round sphere S^2, where geodesics

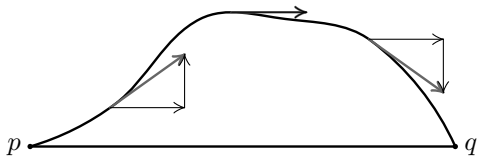


Figure 1.15. Decomposing tangent vectors to show that a straight line is the shortest smooth curve between two points.

are great circles. While the first fact is an article of faith in elementary geometry, it requires a proof using a certain amount of calculus. We will sketch the proof, but for a reader not familiar with calculations involving arbitrary curves, we recommend carrying out the argument in detail as an exercise.

Consider an arbitrary parametrised curve with endpoints p and q, and project it to the straight line pq. As a parametrised curve, the projection is no longer than the original one—in fact, it is strictly shorter if the original curve does not lie entirely on the line.

If the curve is smooth, this follows from the formula for the length of the curve as the integral of the length of its tangent vector, which decomposes into two components, one parallel to the line pq, and one perpendicular (Figure 1.15). For an arbitrary curve, one can use an approximation by a polygonal curve—in either case, having established that the length of the original curve is greater than or equal to the length of the projected curve, one uses integration to show that the length of the projected curve is greater than or equal to the length of the interval pq, with equality if and only if the parameter is monotone (so that the curve is a reparametrised interval).

A very similar argument can be carried out on the sphere, using geographic coordinates around the point p and projection along parallels to the meridian (great circle) passing through p and q. In fact, once it is understood just what is needed for this argument, it can be adapted in many cases to find geodesics.

It is sometimes the case that one can find geodesics on other surfaces by reducing the question to a known situation. For example, the following exercise can be solved by reducing the question to the case of the Euclidean plane.

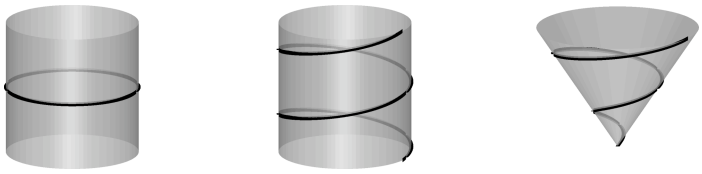


Figure 1.16. Three curves in $\mathbb{R}^3$.

Exercise 1.11. Find all geodesics on the round cylinder

$$\{\,(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2=1\,\}$$

and the upper half of the round cone

$$\{\,(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2-z^2=0,\ z\geq 0\,\}.$$

c. Parametric representations of curves. We often write a curve in $\mathbb{R}^2$ as the solution of a particular equation; the unit circle, for example, is the set of points satisfying $x^2+y^2=1$. This implicit representation becomes more difficult in higher dimensions; in general, each equation we require the coordinates to satisfy will remove a degree of freedom (assuming independence) and hence a dimension, so to determine a curve in $\mathbb{R}^3$ we require not one, but two equations. Geometrically, we are obtaining a curve as the intersection of two surfaces, each specified by one of the equations. For example, the unit circle lying in the xy-plane is the solution set of

$$\begin{aligned} x^2+y^2&=1\\ z&=0 \end{aligned}$$

which is the intersection of this plane with a cylinder of unit radius. This is a simple example, for which these equations and the visualisation of the surfaces pose no real difficulty; there are many examples which are more difficult to deal with in this manner, but which can be easily written down using a *parametric representation*. That is, we define the curve in question as the set of all points given by

$$(x,y,z)=(f_1(t),f_2(t),f_3(t))$$

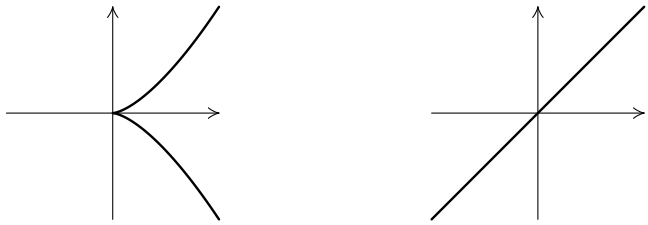


Figure 1.17. Two curves with a vanishing tangent vector at $t = 0$.

where t lies in the interval $[a, b]$, whose endpoints a and b may be $\pm\infty$. In this representation, the circle discussed above would be written

$$(x, y, z) = (\cos t, \sin t, 0)$$

with $0 \leq t \leq 2\pi$. If we replace the equation $z = 0$ with $z = t$, we obtain not a circle, but a helix; it takes a little more imagination to picture this as the intersection of two surfaces. We could also multiply the expressions for x and y by t to describe a spiral on the cone, whose implicit representation is again not immediate.

Exercise 1.12. Find two equations whose common solution set is the helix.

If we expect our curve to be smooth, we must impose certain conditions on the coordinate functions f_i. The first condition is that each f_i be continuously differentiable; this will guarantee the existence of a continuously varying tangent vector at every point along the curve. However, if we do not impose the further requirement that this tangent vector be nonvanishing, that is, that $(f_1')^2 + (f_2')^2 + (f_3')^2 \neq 0$ holds everywhere on the curve, then the curve may still fail to be smooth.

As a simple but important example of what may happen when this condition is violated, consider the curve $(x, y) = (t^2, t^3)$. The tangent vector $(2t, 3t^2)$ vanishes at $t = 0$, which appears as a *cusp* at the origin in Figure 1.17. So in this case, even though f_1 and f_2 are perfectly smooth functions, the curve itself is not smooth.

The nonvanishing condition is sufficient, but not necessary, to have a smooth curve; to see the latter, consider the curve $x = t^3$, $y = t^3$. The tangent vector vanishes when $t = 0$, but the curve itself is just the line $x = y$, which is as smooth as we could possibly ask for. In this case we could reparametrise the curve to obtain a parametric representation in which the tangent vector is everywhere nonvanishing.

d. Difficulties with representation by embedding. Parametric representations of curves (and surfaces as well), along with representations as level sets of functions (the implicit representations we saw before) all embed the curve or surface into an ambient Euclidean space, which so far has usually been $\mathbb{R}^3$. Our subsequent dealings have sometimes relied on properties of this ambient space; for example, the usual definition of the length of a curve relies on a broken line approach, in which the curve is approximated by a piecewise linear 'curve', whose length we can compute using the usual notion of Euclidean distance.

What happens, though, if our surface does not live in $\mathbb{R}^3$? We already touched upon this problem in Lecture 1(b), and now return to it in more depth, as $\mathbb{R}^3$ is not the proper setting for several of the surfaces we have seen so far. For example, $\mathbb{R}P^2$ cannot be embedded in $\mathbb{R}^3$, so if we are to compute the length of curves in the elliptic plane, we must either embed it in $\mathbb{R}^4$ or some higher dimensional space, or else come up with a new definition of length, an issue to which we shall return in Lecture 23.

Our discussion of factor spaces in Lecture 1 was motivated by the example of the Klein bottle, which was defined as a factor space of the square, or rectangle, where the left and right edges are identified with direction reversed (as with the Möbius strip), but in addition, the top and bottom edges are identified (without reversing direction). We mentioned then that the Klein bottle cannot be embedded into $\mathbb{R}^3$, and that the closest one can come is to imagine rolling the square into a cylinder, then attaching the ends of the cylinder after passing one end through the wall of the cylinder into the interior.

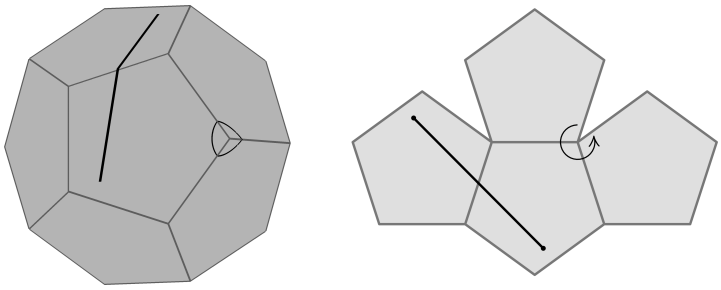


Figure 1.18. Life on a dodecahedron.

Of course, this results in the surface intersecting itself in a circle; in order to avoid this self-intersection, we could add a dimension and embed the surface in $\mathbb{R}^4$. Given the extra dimension to work with, we could begin with the immersion described above and perform the four-dimensional analogue of taking a string which is lying in a figure eight on a table, and lifting part of it off the surface of a table in order to avoid having it touch itself. No such manoeuvre is possible for the Klein bottle in three dimensions, but the immersion of the Klein bottle into $\mathbb{R}^3$ is still a popular shape, and some enterprising craftsman has been selling both 'Klein bottles' and beer mugs in the shape of Klein bottles at the yearly meetings of the American Mathematical Society. We had two such glass models of Klein bottles in the class, which were bought there: one is a conventional inverted bottle very similar to the image in Figure 1.6; the other is a "Klein beer mug", very close to a usual one in its outside shape and usable as a drinking vessel.

Even when an embedding exists, it is possible for the choice of embedding to obscure certain geometric properties of an object. Consider the surface of a dodecahedron (or any solid, for that matter). From the point of view of the embedding in $\mathbb{R}^3$, there are three sorts of points on the surface; a given point can lie either at a vertex, along an edge, or on a face. Being three-dimensional creatures, we see these as three distinct classes of points.

Now imagine that we are two-dimensional creatures living on the surface of the dodecahedron. We can tell whether or not we are at

a vertex; at a vertex, the angles add up to less than 2π, whereas everywhere else, they add up to exactly 2π. However, we cannot tell whether or not we are at an edge; this has to do with the fact that given two points on adjacent faces, the way to find the shortest path between them is to unfold the two faces and place them flat on the plane (at which stage points on an edge look just like points on a face), draw a straight line between the two points in question, and then fold the surface back up (Figure 1.18). As far as our two-dimensional selves are concerned, points on an edge and points on a face are indistinguishable, since the unfolding process does not change any distances along the surface.

It is also possible that a surface which can be embedded in $\mathbb{R}^3$ will lose some of its nicer properties in the process. For example, the usual embedding of the torus destroys the symmetry between meridians and parallels; all of the meridians are the same length, but the length of the parallels varies. We can retain this symmetry by embedding in $\mathbb{R}^4$, the so-called *flat torus*. Parametrically, this is given by

$$\begin{aligned} x &= r\cos t & y &= r\sin t \\ z &= r\cos s & w &= r\sin s \end{aligned}$$

where $s, t \in [0, 2\pi]$. As we already mentioned, we can also obtain the flat torus as a factor space, using the same method as in the definition of the projective plane or Klein bottle. Beginning with a rectangle, we identify opposite sides (with no reversal of direction); alternately, we can consider the family of isometries of $\mathbb{R}^2$ given by $T_{m,n}\colon (x, y) \mapsto (x + m, y + n)$, where $m, n \in \mathbb{Z}$, and mod out by orbits. This construction of $\mathbb{T}^2$ as $\mathbb{R}^2/\mathbb{Z}^2$ is exactly analogous to the construction of the circle S^1 as $\mathbb{R}/\mathbb{Z}$.

We have seen that surfaces can be considered from different viewpoints: sometimes we treat them as geometric objects, with intrinsically defined distances, angles, and areas, while other times we treat them as 'stretchable' objects which can be bent and deformed, but not torn or broken. In mathematical language, this corresponds to considering different structures on surfaces, and this is the central theme of this course, which we will take up in earnest in the next lecture.

Before doing so, we would like to fix a linguistic ambiguity; for example, what should the word 'sphere' mean? How will we indicate whether we are treating a particular surface as a geometric object, or as a topological one (that is, one which may be deformed without changing the nature of the surface)? Our convention will be as follows: an indefinite article in front of the name, as in 'a sphere', 'a torus' or 'a projective plane', will mean that we consider the object in the topological sense, up to a homeomorphism. The use of an adjective or the definite article will generally signify a smaller class of objects, as in 'a sphere given by an equation'. Then 'a round sphere' would mean any sphere which has 'spherical geometry', that is, which is isometric to the actual sphere in Euclidean space. Similarly, 'a flat torus' signifies any torus with locally Euclidean geometry, while 'the flat torus' or 'the torus' will indicate the unit square with opposite sides identified, endowed with the appropriate geometry inherited from $\mathbb{R}^2$; sometimes we will call this object 'the standard flat torus'. 'The elliptic plane' indicates the factor space of the unit sphere in which antipodal points are identified, with geometry inherited from the sphere, and so on for various other examples which will arise.

Exercise 1.13. Write parametric representations for a projective plane in each of the following:

(1) $\mathbb{R}^3$ (with self-intersections).

(2) $\mathbb{R}^4$ (without self-intersections).

e. Regularity conditions for parametrically defined surfaces. A parametrisation of a surface in $\mathbb{R}^3$ is given by a region $U \subset \mathbb{R}^2$ with coordinates $(t, s) \in U$ and a set of three maps f_1, f_2, f_3; the surface is then the image of $F = (f_1, f_2, f_3)$, the set of all points $(x, y, z) = (f_1(t, s), f_2(t, s), f_3(t, s))$.

As with parametric representations of curves, we need a regularity condition to ensure that our surface is in fact smooth, without cusps or singularities. We once again require that the functions f_i be continuously differentiable, but now it is insufficient to simply require that the matrix of derivatives Df be nonzero. Rather, we require that

it have maximal rank; the matrix is given by

$$Df = \begin{pmatrix} \partial_s f_1 & \partial_t f_1 \\ \partial_s f_2 & \partial_t f_2 \\ \partial_s f_3 & \partial_t f_3 \end{pmatrix}$$

and so our requirement is that the two tangent vectors to the surface, given by the columns of Df, be linearly independent. Under this condition, the Implicit Function Theorem guarantees that the parametric representation is locally bijective and that its inverse is differentiable.

Parametric representations may of course have singularities. A good example is the representation of the sphere given by the inverse map to the geographic coordinates, which maps an open disc regularly onto the sphere with a point removed, and collapses the boundary of the disc into this single point.

Lecture 4.

a. Remarks on metric spaces and topology. Geometry in its most immediate form deals with measuring distances.[8] For this reason, *metric spaces* are fundamental objects in the study of geometry. In the geometric context, the distance function itself is the object of interest; this stands in contrast to the situation in analysis, where metric spaces are still fundamental (as spaces of functions, for example), but where the metric is introduced primarily in order to have a notion of convergence, and so the *topology* induced by the metric is the primary object of interest, while the metric itself stands somewhat in the background.

A metric space is a set X, together with a metric, or distance function, $d\colon X \times X \to \mathbb{R}_0^+$, which satisfies the following axioms for all values of the arguments:

(1) Positivity: $d(x, y) \geq 0$, with equality iff $x = y$

(2) Symmetry: $d(x, y) = d(y, x)$

(3) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$

[8]The reader should be aware, however, that in modern mathematical terminology, the word 'geometry' may appear with adjectives like 'affine' or 'projective'. Those branches of geometry study structures which do not involve distances directly.

The last of these is generally the most interesting, and is sometimes useful in the following equivalent form:

$$d(x, y) \geq |d(x, z) - d(y, z)|$$

Once we have defined a metric on a space X, we immediately have a topology on X induced by that metric. The *ball* in X with centre x and radius r is given by

$$B(x, r) = \{\, y \in X \mid d(x, y) < r \,\}$$

Then a set $A \subset X$ is said to be *open* if for every $x \in A$, there exists $r > 0$ such that $B(x, r) \subset A$, and A is *closed* if its complement $X \setminus A$ is open. We now have two equivalent notions of convergence: in the metric sense, $x_n \to x$ if $d(x_n, x) \to 0$, while the topological definition requires that for every open set U containing x, there exist some N such that for every $n > N$, we have $x_n \in U$. It is not hard to see that these are equivalent.

Similarly for the definition of continuity; we say that a function $f\colon X \to Y$ is *continuous* if $x_n \to x$ implies $f(x_n) \to f(x)$. The equivalent definition in more topological language is that continuity requires $f^{-1}(U) \subset X$ to be open whenever $U \subset Y$ is open. We say that f is a *homeomorphism* if it is a bijection and if both f and f^{-1} are continuous.

Exercise 1.14. Show that the two sets of definitions (metric and topological) in the previous two paragraphs are equivalent.

Within mathematics, there are two broad categories of concepts and definitions with which we are concerned. In the first instance, we seek to fully describe and understand a particular sort of structure. We make a particular definition or construction, and then seek to either show that there is only one object (up to some appropriate notion of isomorphism) which fits our definition, or to give some sort of classification which exhausts all the possibilities. Examples of this approach include Euclidean space, which is unique once we specify dimension, or Jordan normal form, which is unique for a given matrix up to a permutation of the basis vectors, as well as finite simple groups, or semi-simple Lie algebras, for which we can (eventually) obtain a complete classification.

No such uniqueness or classification result is possible with metric spaces and topological spaces in general; these definitions are examples of the second sort of mathematical object, and are generalities rather than specifics. In and of themselves, they are far too general to allow any sort of complete classification or universal understanding, but they have enough properties to allow us to eliminate much of the tedious case by case analysis which would otherwise be necessary when proving facts about the objects in which we are really interested. The general notion of a group, or of a Banach space, also falls into this category of generalities.

Before moving on, there are three definitions of which we ought to remind ourselves. First, recall that a metric space is *complete* if every Cauchy sequence converges. This is not a purely topological property, since we need a metric in order to define Cauchy sequences; to illustrate this fact, notice that the open interval $(0, 1)$ and the real line $\mathbb{R}$ are homeomorphic, but that the former is not complete, while the latter is.

Secondly, we say that a metric space (or subset thereof) is *compact* if every sequence has a convergent subsequence. In the context of general topological spaces, this property is known as sequential compactness, and the definition of compactness is given as the requirement that every open cover have a finite subcover; for our purposes, since we will be dealing with metric spaces, the two definitions are equivalent. There is also a notion of *precompactness*, which requires every sequence to have a *Cauchy* subsequence.

The knowledge that X is compact allows us to draw a number of conclusions; the most commonly used one is that every continuous function $f\colon X \to \mathbb{R}$ is bounded, and in fact achieves its maximum and minimum. In particular, the product space $X \times X$ is compact, and so the distance function is bounded.

Finally, we say that X is *connected* if it cannot be written as the union of non-empty disjoint open sets; that is, $X = A \cup B$, A and B open, $A \cap B = \emptyset$ implies either $A = X$ or $B = X$. There is also a notion of *path connectedness*, which requires for any two points $x, y \in X$ the existence of a continuous function $f\colon [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. As is the case with the two forms of

compactness above, these are not equivalent for arbitrary topological spaces (or even for arbitrary metric spaces—the usual counterexample is the union of the graph of $\sin(1/x)$ with the vertical axis), but will be equivalent on the class of spaces with which we are concerned.

b. Homeomorphisms and isometries. In the topological context, the natural notion of equivalence between two spaces is that of homeomorphism, which we defined above as a continuous bijection with continuous inverse. Two topological spaces are *homeomorphic* if there exists a homeomorphism between them. Any property common to all homeomorphic spaces is called a *topological invariant*; this naturally includes any property defined in purely topological terms, such as connectedness, path-connectedness, and compactness.

Some invariants require a little more work; for example, we would like to believe that dimension is a topological invariant, and this is in fact true,[9] but proving that $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic for $m \neq n$ requires non-trivial tools.

A considerable part of this course deals with topological invariants of compact surfaces, and in particular, the task of classifying such surfaces up to a homeomorphism. We will almost succeed in solving this problem completely; the only assumption we will have to make is that the surfaces in question admit one of several natural additional structures. In fact this assumption turns out to be true for any surface, but we do not prove this in this course.

The natural equivalence relation in the geometric setting is isometry; a map $f\colon X \to Y$ between metric spaces is *isometric* if

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$$

for every $x_1, x_2 \in X$. If in addition f is a bijection, we say f is an *isometry*. We are particularly interested in the set of isometries from X to itself,

$$\mathrm{Isom}(X, d) = \{\, f\colon X \to X \mid f \text{ is an isometry} \,\}$$

which we can think of as the symmetries of X. In general, the more symmetric X is, the larger this set.

[9] At least for the usual definition of dimension; we mention an alternate definition in the next section.

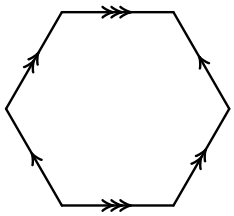


Figure 1.19. A planar model on a hexagon.

In fact, $\mathrm{Isom}(X, d)$ is not just a set; it has a natural binary operation given by composition, under which is becomes a group. This is an example of a very natural and general sort of group which is often of interest; all the bijections from some fixed set to itself, with composition as the group operation. On a finite set, this gives the symmetric group S_n, the group of permutations. On an infinite set, the group of all bijections becomes somewhat unwieldy, and it is more natural to consider the subgroup of bijections which preserve a particular structure, in this case the metric structure of the space. Another common example of this is the general linear group $GL(n, \mathbb{R})$, which is the group of all bijections from $\mathbb{R}^n$ to itself preserving the linear structure of the space.

In the next lecture, we will discuss the isometry groups of Euclidean space and of the sphere.

Exercise 1.15. Consider a regular hexagon with pairs of opposite sides identified by the corresponding translations, as in Figure 1.19.

(1) Prove that it is a torus.

(2) Prove that locally, it is isometric to Euclidean plane.

(3) Prove that it is not isometric to the standard flat torus.

c. Other notions of dimension. As mentioned above, we usually think of dimension as a topological invariant. However, for general compact metric spaces there is another notion of dimension which is a metric invariant, rather than a topological one. The main idea is to capture the rate at which volume (or some other sort of measure) scales with the metric; for example, a cube in $\mathbb{R}^n$ with side length r has volume r^n, and the exponent n is the dimension of the space.

In general, given a compact metric space X, for any $\varepsilon > 0$, let $N(\varepsilon)$ be the minimum number of ε-balls required to cover X; that is, the minimum number of points $x_1, \ldots, x_{N(\varepsilon)}$ in X such that every point in X lies within ε of some x_i. This may be thought of as measuring the average 'volume' of an ε-ball, in some sense; the *upper box dimension* of X is defined to be

$$\bar{d}_{\mathrm{box}}(X) = \limsup_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log 1/\varepsilon}.$$

We take the upper limit because the limit itself may not exist. The *lower box dimension* is defined similarly, taking the lower limit instead. These notions of dimension do not behave quite as nicely as we would like in all situations; for example, the set of rational numbers, which is countable, has upper and lower box dimension equal to one.

There is a more effective notion of *Hausdorff dimension*, which eliminates the need to distinguish between upper and lower limits, and which is equal to zero for any countable set; because its definition requires an understanding of measure theory, we will not discuss it here. For 'good' sets all three definitions coincide, and are central to the study of fractal geometry; however, they are not topological invariants, so our claim in the last section must be understood to apply only to a strictly topological notion of dimension.

d. Geodesics. When we are interested in a metric space as a geometric object, rather than as something in analysis or topology, it is of particular interest to examine those triples (x, y, z) for which the triangle inequality becomes degenerate, that is, for which $d(x, z) = d(x, y) + d(y, z)$.

For example, if our space X is just the Euclidean plane $\mathbb{R}^2$ with distance function given by Pythagoras' formula,

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$$

then the triangle inequality is a consequence of the Cauchy-Schwarz inequality, and we have equality in the one iff we have equality in the other; this occurs iff y lies in the line segment $[x, z]$, so that the three points x, y, z are in fact collinear.

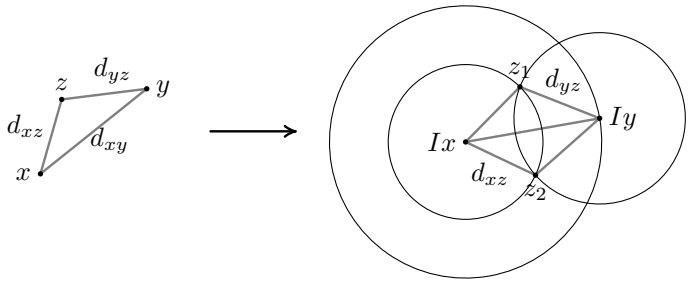


Figure 1.20. Images of three points determine an isometry.

A similar observation holds on the sphere, where the triangle inequality becomes degenerate for the triple (x, y, z) iff y lies along the shorter arc of the great circle connecting x and z. So in both these cases, degeneracy occurs when the points lie along a geodesic; this suggests that in general, a characteristic property of a geodesic is the relation $d(x, z) = d(x, y) + d(y, z)$ whenever y lies between two points x and z which are sufficiently close along the curve.

Lecture 5.

a. Isometries of the Euclidean plane. There are three ways to describe and study isometries of the Euclidean plane: synthetic; as affine maps in two real dimensions; and as affine maps in one complex dimension. The last two methods are closely related. We begin with observations using the traditional synthetic approach.

If we fix three noncollinear points in $\mathbb{R}^2$ and want to describe the location of a fourth, it is enough to know its distance from each of the first three. This may readily be seen from the fact that three circles whose centres are not collinear intersect in at most one point.

As a consequence of this, an isometry of $\mathbb{R}^2$ is completely determined by its action on three noncollinear points. In fact, if we have an isometry $I\colon \mathbb{R}^2 \to \mathbb{R}^2$, and three such points x, y, z, as in Figure 1.20, the choice of Ix constrains Iy to lie on the circle with centre Ix and radius $d(x, y)$, and once we have chosen Iy, there are only two possibilities for Iz; one (z_1) corresponds to the case where I preserves orientation, the other (z_2) to the case where orientation is

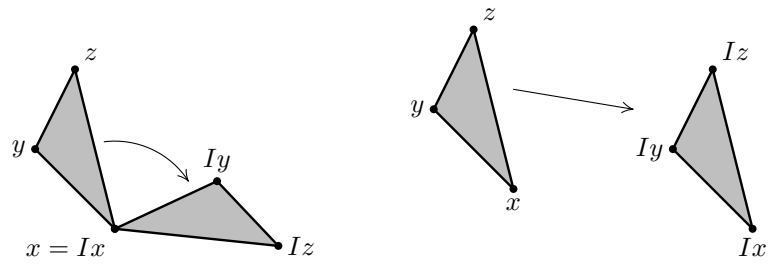


Figure 1.21. Orientation preserving isometries.

reversed. So for two pairs of distinct points a, b and a', b' such that the distances between a and b and between a' and b' coincide, there are exactly two isometries which map a to a' and b to b'; one of these will be orientation preserving, the other orientation reversing.

Passing to algebraic descriptions, notice that any isometry I must carry lines to lines, since as we saw last time, three points in the plane are collinear iff the triangle inequality becomes degenerate. Thus it is an *affine map*—that is, a composition of a linear map and a translation—so it may be written as $I\colon x \mapsto Ax + b$, where $b \in \mathbb{R}^2$ and A is a 2×2 matrix. In fact, A must be orthogonal, which means that we can write things in terms of the complex plane $\mathbb{C}$ and get (in the orientation preserving case) $I\colon z \mapsto az + b$, where $a, b \in \mathbb{C}$ and $|a| = 1$. In the orientation reversing case, we have $I\colon z \mapsto a\bar{z} + b$.

Using the preceding discussion, we can now classify any isometry of the Euclidean plane as belonging to one of four types, depending on whether it preserves or reverses orientation, and whether or not it has a fixed point.

Case 1: An orientation preserving isometry which possesses a fixed point is a *rotation*. Let x be the fixed point, $Ix = x$. Fix another point y; both y and Iy lie on a circle of radius $d(x, y)$ around x. The rotation about x which takes y to Iy satisfies these criteria, which are enough to uniquely determine I given that it preserves orientation, hence I is exactly this rotation.

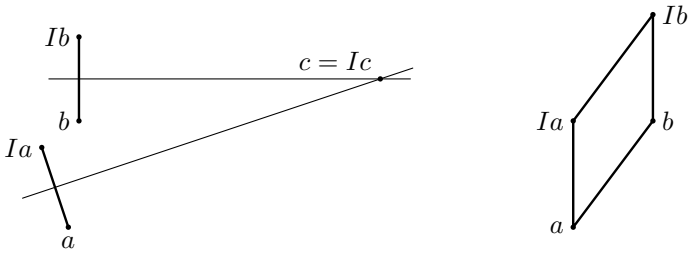


Figure 1.22. An orientation preserving isometry with no fixed points is a translation.

Rotations are entirely determined by the centre of rotation and the angle of rotation, so we require three parameters to specify a rotation.

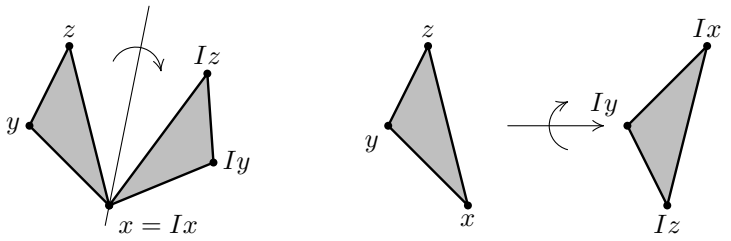
Case 2: An orientation preserving isometry I with no fixed points is a *translation*. The easiest way to see that is to use the complex algebraic description. Writing $Iz = az + b$ with $|a| = 1$, we observe that if $a \neq 1$, we can solve $az + b = z$ to find a fixed point for I. Since no such point exists, we have $a = 1$, hence $I\colon z \mapsto z + b$ is a translation.

One can also make a purely synthetic argument for this case; we show that the intervals $[a, Ia]$ and $[b, Ib]$ must be parallel and of equal length for every a, b. Indeed, if they fail to be parallel for some a, b, then their perpendicular bisectors intersect in some point c, as shown in Figure 1.22. Since $[a, Ia, c]$ and $[b, Ib, c]$ are isosceles triangles, we have $d(a, c) = d(Ia, c)$ and $d(b, c) = d(Ib, c)$, hence $Ic = c$ since I preserves orientations.

But I has no fixed point, and so $[a, Ia]$ and $[b, Ib]$ must be parallel; since I is an isometry, $d(Ia, Ib) = d(a, b)$, and hence the quadrilateral $[a, Ia, Ib, b]$ is a parallelogram. It follows that the intervals $[a, Ia]$ are all parallel and of equal length, and so I is a translation.

We only require two parameters to specify a translation; since the space of translations is two-dimensional, almost every orientation preserving isometry is a rotation, and hence has a fixed point.

Case 3: An orientation reversing isometry which possesses a fixed point is a *reflection*. Say $Ix = x$, and fix $y \neq x$. Let ℓ be the line



Reflection—a line of fixed points Glide reflection—no fixed points

Figure 1.23. Orientation reversing isometries.

bisecting the angle formed by the points y, x, Iy. Using the same approach as in case 1, the reflection through ℓ takes x to Ix and y to Iy; since it reverses orientation, I is exactly this reflection.

It takes two parameters to specify a line, and hence a reflection, so the space of reflections is two-dimensional.

Case 4: An orientation reversing isometry with no fixed point is a *glide reflection*. Let T be the unique translation that takes x to Ix. Then $I = R \circ T$ where $R = I \circ T^{-1}$ is an orientation reversing isometry which fixes Ix. By the above, R must be a reflection through some line ℓ. Decompose T as $T_1 \circ T_2$, where T_1 is a translation by a vector perpendicular to ℓ, and T_2 is a translation by a vector parallel to ℓ. Then $I = R \circ T_1 \circ T_2$, and $R \circ T_1$ is reflection through a line parallel to ℓ, hence I is the composition of a translation T_2 and a reflection $R \circ T_1$ which commute; that is, a glide reflection.

A glide reflection is specified by three parameters; hence the space of glide reflections is three-dimensional, so almost every orientation reversing isometry is a glide reflection, and hence has no fixed point.

The group $\mathrm{Isom}(\mathbb{R}^2)$ is a topological group with two components; one component comprises the orientation preserving isometries, the other the orientation reversing isometries. From the above discussions of how many parameters are needed to specify an isometry, we see that the group is three-dimensional; in fact, it has a nice embedding

into the group $GL(3,\mathbb{R})$ of invertible 3×3 matrices:

$$\mathrm{Isom}(\mathbb{R}^2) = \left\{ \begin{pmatrix} O(2) & \mathbb{R}^2 \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} \mathbb{R}^2 \\ 1 \end{pmatrix} \to \begin{pmatrix} \mathbb{R}^2 \\ 1 \end{pmatrix} \right\}.$$

Here $O(2)$ is the group of real valued orthogonal 2×2 matrices, and the plane upon which $\mathrm{Isom}(\mathbb{R}^2)$ acts is the horizontal plane $z=1$ in $\mathbb{R}^3$.

Exercise 1.16. Prove that every isometry of the Euclidean plane can be represented as a product of at most three reflections.

Exercise 1.17. Consider all possible configurations of two and three lines in the plane: two lines may be either parallel or intersecting; for three lines there are a few more options. Identify the product of reflections in those lines for each case as one of four types of isometries.

Exercise 1.18. Consider an orientation reversing isometry in the complex form $z \mapsto a\bar{z} + b$. Find a condition on $a, b \in \mathbb{C}$ which will determine if it is a reflection or a glide reflection, and identify the axis in both cases.

b. Isometries of the sphere and the elliptic plane. By counting dimensions in the isometry group of the Euclidean plane, we argued that almost every orientation preserving isometry has a fixed point, while almost every orientation reversing isometry has no fixed point. In the next lecture, we will see that the picture for the sphere is somewhat similar—now *any* orientation preserving isometry has a fixed point, and most orientation reversing ones have none. For the elliptic plane, however, it will turn out to be dramatically different: *any isometry has a fixed point*, and can in fact be interpreted as a rotation!

Many of the arguments in the previous section carry over to the sphere; the same techniques of taking intersections of circles, etc. still apply. The classification of isometries on the sphere is somewhat simpler, since every orientation preserving isometry has a fixed point, while every orientation reversing isometry (other than reflection in a great circle) has a point of period two, which becomes a fixed point when we pass to the elliptic plane.

We will be able to show that every orientation preserving isometry of the sphere comes from a rotation of $\mathbb{R}^3$, and that the product of two rotations is itself a rotation. This is slightly different from the case with $\mathrm{Isom}(\mathbb{R}^2)$, where the product could either be a rotation, or if the two angles of rotation summed to zero (or a multiple of 2π), a translation. We will, in fact, be able to obtain $\mathrm{Isom}(S^2)$ as a group of 3×3 matrices in a much more natural way than we did for $\mathrm{Isom}(\mathbb{R}^2)$ above, since any isometry of S^2 extends to a linear orthogonal map of $\mathbb{R}^3$, and so we will be able to use linear algebra directly.

Lecture 6.

a. Classification of isometries of the sphere and the elliptic plane. There are two approaches we can take to investigating isometries of the sphere S^2; we saw this dichotomy begin to appear when we examined $\mathrm{Isom}(\mathbb{R}^2)$. The first is the *synthetic* approach, which treats the problem using the tools of solid geometry; this is the approach used by the Greek geometers of late antiquity in developing spherical geometry for use in astronomy.

The second approach, which we will follow below, uses methods of linear algebra; translating the question about geometry to a question about matrices puts a wide range of techniques at our disposal, which will prove enlightening, and rather more useful now than it was in the case of the plane, when the relevant matrices were only 2×2.

The first important result is that there is a natural bijection (which is in fact a group isomorphism) between $\mathrm{Isom}(S^2)$ and $O(3)$, the group of real orthogonal 3×3 matrices. The latter is defined by

$$O(3) = \{\, A \in M_3(\mathbb{R}) \mid A^T A = I \,\}$$

That is, $O(3)$ comprises those matrices for which the transpose and the inverse coincide. This has a nice geometric interpretation; we can think of the columns of a 3×3 matrix as vectors in $\mathbb{R}^3$, so that $A = (a_1|a_2|a_3)$, where $a_i \in \mathbb{R}^3$. (In fact, a_i is the image of the i^{th} basis vector e_i under the action of A). Then A lies in $O(3)$ iff $\{a_1, a_2, a_3\}$ forms an orthonormal basis for $\mathbb{R}^3$, that is, if $\langle a_i, a_j \rangle = \delta_{ij}$, where $\langle \cdot, \cdot \rangle$ denotes inner product, and δ_{ij} is the Kronecker delta, which takes the

value 1 if $i = j$, and 0 otherwise. The same criterion applies if we consider the rows of A, rather than the columns.

Since $\det(A^T) = \det(A)$, any matrix $A \in O(3)$ has determinant ± 1; the sign of the determinant indicates whether the map preserves or reverses orientation. The group of real orthogonal matrices with determinant equal to positive one is the *special orthogonal group* $SO(3)$.

In order to see that the members of $O(3)$ are in fact the isometries of S^2, we could take the synthetic approach and look at the images of three points not all lying on the same geodesic, as we did with $\mathrm{Isom}(\mathbb{R}^2)$; in particular, the standard basis vectors e_1, e_2, e_3.

An alternate approach is to extend the isometry to $\mathbb{R}^3$ by homogeneity. That is, given an isometry $I\colon S^2 \to S^2$, we can define a linear map $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$Ax = \|x\| \cdot I\left(\frac{x}{\|x\|}\right)$$

It follows that A preserves lengths in $\mathbb{R}^3$, and in fact, this is sufficient to show that it preserves angles as well. This can be seen using a technique called *polarisation*, which allows us to express the inner product in terms of the norm, and hence show the general result that preservation of norm implies preservation of inner product:

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle \\ &= \|x\|^2 + \|y\|^2 + 2\langle x,y\rangle \\ \langle x,y\rangle &= \frac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2) \end{aligned}$$

This is a useful trick to remember, and allows us to show that a symmetric bilinear form is determined by its diagonal part. In our particular case, it shows that the matrix A we obtained is in fact in $O(3)$, since it preserves both lengths and angles.

The matrix $A \in O(3)$ has three eigenvalues, some of which may be complex. Because A is orthogonal, we have $|\lambda| = 1$ for each eigenvalue λ; further, because the determinant is the product of the eigenvalues, we have $\lambda_1\lambda_2\lambda_3 = \pm 1$. The entries of the matrix A are real, hence the

coefficients of the characteristic polynomial are as well; this implies that if λ is an eigenvalue, so is its complex conjugate $\bar{\lambda}$.

There are two cases to consider. Suppose $\det(A) = 1$. Then the eigenvalues are $\lambda, \bar{\lambda}$, and 1, where $\lambda = e^{i\alpha}$ lies on the unit circle in the complex plane. Let x be the eigenvector corresponding to the eigenvalue 1, and note that A acts on the plane orthogonal to x by rotation by α; hence A is a rotation by α around the axis through x.

The second case, $\det(A) = -1$, can be dealt with by noting that A can be written as a composition of $-I$ (reflection through the origin) with a matrix with positive determinant, which must be a rotation, by the above discussion. Upon passing to the elliptic plane $\mathbb{R}P^2$, the reflection $-I$ becomes the identity, so that *every* isometry of $\mathbb{R}P^2$ is a rotation.

This result, that every isometry of the sphere is either a rotation or the composition of a rotation and a reflection through the origin, shows that every isometry has either a fixed point or a point of period two, which becomes a fixed point upon passing to the quotient space $\mathbb{R}P^2$.

As an concrete example of how all isometries become rotations in $\mathbb{R}P^2$, consider the map A given by reflection through the xy-plane, $A(x, y, z) = (x, y, -z)$. Let R be rotation by π about the z-axis, given by $R(x, y, z) = (-x, -y, z)$. Then $A = R \circ (-I)$, so that as maps on $\mathbb{R}P^2$, A and R coincide. Further, any point $(x, y, 0)$ on the equator of the sphere is fixed by this map, so that R fixes not only one point in $\mathbb{R}P^2$, but many.

Exercise 1.19. Let x and y be two points in the elliptic plane.

(1) Prove that there are at most two shortest curves connecting x and y.

(2) Find a necessary and sufficient condition for uniqueness of the shortest curve connecting x and y.

b. Area of a spherical triangle. In the Euclidean plane, the most symmetric formula for determining the area of a triangle is Heron's formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

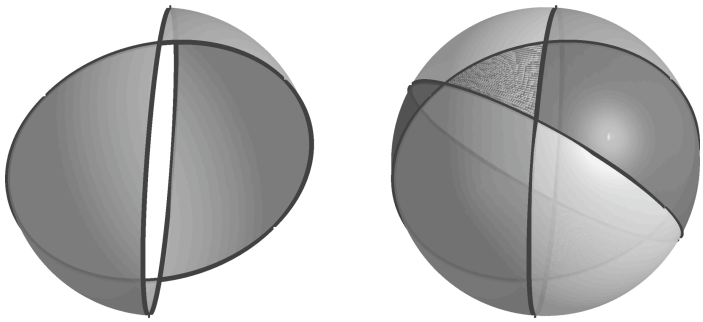


Figure 1.24. Determining the area of a spherical triangle.

where a, b, c are the lengths of the sides, and $s = \frac{1}{2}(a + b + c)$ is the semiperimeter of the triangle. There are other, less symmetric, formulas available to us if we know the lengths of two sides and the measure of the angle between them, or two angles and a side; if all we have are the angles, however, we cannot determine the area, since the triangle could be scaled up or down, preserving the angles while changing the area.

This is not the case on the surface of the sphere; given a spherical triangle, that is, the area on the sphere enclosed by three geodesics (great circles), we can find the area of the triangle via a wonderfully elegant formula in terms of the angles, as follows.

Consider the 'wedge' lying between two lines of longitude on the surface of a sphere, with an angle α between them. The area of this wedge is proportional to α, and since the surface area of the sphere with radius R is $4\pi R^2$, it follows that the area of the wedge is $\frac{\alpha}{2\pi} 4\pi R^2 = 2\alpha R^2$. If we take this together with its mirror image (upon reflection through the origin), which lies on the other side of the sphere, runs between the same poles, and has the same area, then the area of the 'double wedge' shown in Figure 1.24 is $4\alpha R^2$.

Now consider a spherical triangle with angles α, β, and γ. Put the vertex with angle α at the north pole, and consider the double wedge lying between the two great circles which form the angle α. Paint this double wedge red; as we saw above, it has area $4\alpha R^2$.

Repeat this process with the angle β, painting the new double wedge yellow, and with γ, painting that double wedge blue. Now every point on the sphere has been painted exactly one colour (or, as in Figure 1.24, one particular shade of gray), with the exception of the points lying inside our triangle, and the points diametrically opposite them, which have been painted all three colours. (We neglect the boundaries of the wedges, since they have area zero). Hence if we add up the areas of the double wedges, we obtain

$$\begin{aligned}\sum \text{areas of wedges} &= \text{blue area} + \text{yellow area} + \text{red area} \\ &= (\text{area of sphere}) + 4 \times (\text{area of triangle})\end{aligned}$$

which allows us to write an equation for the area A of the triangle:

$$4(\alpha + \beta + \gamma)R^2 = 4\pi R^2 + 4A$$

Solving, we see that

$$A = R^2(\alpha + \beta + \gamma - \pi). \tag{1.6}$$

Thus the area of the triangle is directly proportional to its *angular excess*; this result has no analogue in planar geometry, due to the flatness of the Euclidean plane. As we will see later on in the course, it does have an analogue in the hyperbolic plane, where the angles of a triangle add up to less than π, and the area is proportional to the *angular defect*.

Exercise 1.20. Express the area of a geodesic polygon on the sphere in terms of its angles.

Lecture 7.

a. Spaces with lots of isometries. In our discussion of the isometries of $\mathbb{R}^2$, S^2, and $\mathbb{R}P^2$, we have observed a number of differences between the various spaces, as well as a number of similarities. One of the most important similarities is the high degree of symmetry each of these spaces possesses, as evidenced by the size of their isometry groups.

We can make this a little more concrete by observing that the isometry group acts *transitively* on each of these spaces; given any

two points a and b in the plane, on the sphere, or in the projective plane, there is an isometry I of the space such that $Ia = b$.

In fact, we can make the stronger observation that the group acts transitively on the set of unit tangent vectors. That is to say, if v is a unit tangent vector at a, which can be thought of as indicating a particular direction along the surface from the point a, and w is a unit tangent vector at b, then not only can we find an isometry that carries a to b, but we can find one that carries v to w.

Another example of a surface with this property is the hyperbolic plane, which will appear in Chapter 4, and has the remarkable property that its isometry group allows not one but three natural representations as a matrix group (or a factor of such a group by its two-element centre).

In fact, these four examples are the only surfaces for which isometries act transitively on unit tangent vectors. There are of course a number of higher-dimensional spaces with this property: Euclidean spaces, spheres, and projective spaces, which are all analogues of their two-dimensional counterparts, immediately come to mind, and there are many more besides.

As an example of a space for which this property fails, consider the flat torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The property holds locally, in the neighbourhood of a point, but does not hold on the entire space. While $\mathrm{Isom}(\mathbb{T}^2)$ acts transitively on points, it does not act transitively on tangent vectors; some directions lie along geodesics which are closed curves, while other directions do not. Another example is given by the cylinder, and examples of a different nature will appear later when we consider the hyperbolic plane and its factors.

What sorts of isometries does $\mathbb{T}^2$ have? We may consider translations $z \mapsto z + z_0$; rotations of $\mathbb{R}^2$, however, will not generally lead to isometries of $\mathbb{T}^2$, since they will usually fail to preserve the lattice $\mathbb{Z}^2$. The rotation by $\pi/2$ about the origin is permissible, as are the flips around the x- and y-axes, and around the line $x = y$.

In general, $\mathbb{Z}^2$ must be mapped to itself or a translation of itself, and so the isometry group is generated by the group of translations, along with the symmetry group of the lattice. The latter group is

simply D_4, the dihedral group on four letters, which arises as the symmetry group of the square.

Exercise 1.21. Describe all the isometries of

(1) the 'hexagonal' torus of Exercise 1.15;

(2) the flat Möbius strip;

(3) the flat Klein bottle, i.e. the square with appropriately identified pairs of opposite sides.

Consider a more general class of examples, which generalise the construction of the flat torus as $\mathbb{R}^2/\mathbb{Z}^2$. Let L be a *lattice* in $\mathbb{R}^2$—that is, a set of vectors of the form $\{\, mu + nv \mid m, n \in \mathbb{Z} \,\}$, where u and v are two fixed linearly independent vectors. We can identify the factor space $\mathbb{R}^2/L$ with the parallelogram

$$\{\, su + tv \mid 0 \leq s, t \leq 1 \,\}$$

with pairs of opposite sides identified by translations.

Exercise 1.22. Show that the following statements hold.

(1) The factor space $\mathbb{R}^2/L$ is homeomorphic to a torus;

(2) $\mathbb{R}^2/L$ has a natural metric which is locally isometric to $\mathbb{R}^2$;

(3) The isometry group acts transitively on $\mathbb{R}^2/L$.

The 'crystallographic restriction' property established in the following exercise aids in the classification of isometries of these tori.

Exercise 1.23. Show that any nontrivial isometry of $\mathbb{R}^2/L$ with a fixed point has period 2, 3, 4, or 6.

b. Symmetric spaces. The discussion of spaces with lots of isometries is related to the notion of a *symmetric space*, which we will now examine more closely. In what follows, we assume certain properties of geodesics which will be formally described (but not proved) later in this course. In particular, we assume that there is a unique geodesic passing through a given point in a given direction, and that there is a unique shortest geodesic connecting any two sufficiently close points.

Of course, all of this assumes the metric on our surface is given in a nice way, as has been the case with all examples considered so far.[10]

Given a point x on a surface X, we define the *geodesic flip* through x, denoted by I_x, as follows. For each geodesic γ passing through x, each point y lying on γ is sent to the point on γ which is the same distance along the geodesic from x as y is, but in the other direction. It is immediate that this map preserves lengths along geodesics through x; it may happen, however, that the distances *between* these geodesics vary, in which case the map would not be isometric.

If the map is indeed isometric on some neighbourhood of x, and if this property holds for the geodesic flip I_x through any point $x \in X$, then we say that X is *locally symmetric*. The classification of such spaces (in any dimension) is one of the triumphs of Lie theory. Notice that the geodesic flip may not be extendable to a globally defined isometry, so the isometry group of a locally symmetric space may be (and sometimes is) quite small. Although we have not yet encountered any such examples, later on (Lecture 31) we will construct the hyperbolic octagon, whose isometry group can be shown to be finite, even though the space is locally symmetric.

Given two nearby points x, y, we can take the point z lying at the midpoint of the geodesic segment connecting them. Then $I_z x = y$. If X is connected (and hence path connected) then any two points can be connected by a finite chain of neighbourhoods where these local isometries are defined. This implies that for any two points in a locally symmetric space, there exists an isometry between small enough neighbourhoods of those points. In other words, locally such a space looks the same near every point.

If for any point $x \in X$ the geodesic flip I_x can be defined not just locally, but globally (that is, extended to the entire surface X), and if it is in fact an isometry of X, then we say X is *globally symmetric*. In this case, the group of isometries $\mathrm{Isom}(X)$ acts transitively on all of X.

[10] These notions of direction and 'nice' metrics, which are rather vague at the moment, will be made more precise when we discuss smooth manifolds and Riemannian metrics in Chapters 3 and 4.

In the previous lecture we discussed a related, but stronger, notion, in which we require $\mathrm{Isom}(X)$ to act transitively not only on points in X, but also on unit tangent vectors. If this holds, then in particular, given any $x \in X$, there is an isometry of X taking some tangent vector at x to its opposite; this isometry must then be the geodesic flip, and so X is globally symmetric. It is *not* the case, however, that every globally symmetric space has this property of transitive action on tangent vectors; the flat torus is one example.

Examples of symmetric spaces are given by $\mathbb{R}^n$, S^n, and $\mathbb{R}P^n$, as well as by their direct products, about which we will say more momentarily. First, notice that the flat torus is symmetric, being the direct product of two symmetric spaces S^1. However, the embedding of the torus into $\mathbb{R}^3$ produces a space which is *not* symmetric, since the isometry group does not act transitively on the points of the surface. In fact, the isometry group of the embedded torus of revolution (the bagel) in $\mathbb{R}^3$ is a finite extension of a one-dimensional group of rotations, while the isometry group of the flat torus is, as we saw last time, a finite extension of a two-dimensional group of translations. Hence the two surfaces are homeomorphic but not isometric.

The flat torus $\mathbb{R}^2/\mathbb{Z}^2$ has no isometric embedding into $\mathbb{R}^3$, but it is isometric to the embedded torus in $\mathbb{R}^4$ given as the zero set of the two equations

$$x_1^2 + x_2^2 = 1$$
$$x_3^2 + x_4^2 = 1.$$

c. Remarks concerning direct products. Given any two sets X and Y, we can define their *direct product*, sometimes called the *Cartesian product*, as the set of all ordered pairs (x, y):

$$X \times Y = \{\, (x, y) \mid x \in X,\ y \in Y \,\}$$

It is very often the case that if X and Y carry an extra structure, such as that of a group, a topological space, or a metric space, then this structure can be carried over to the direct product in a natural way. For example, the direct product of two groups is a group under pointwise multiplication, and the direct product of two topological spaces is a topological space in the product topology.

If X and Y carry metrics d_X and d_Y, then we can put a metric on $X \times Y$ in the same manner as we put a metric on $\mathbb{R}^2$, by defining

$$d((x,y),(x',y')) = \sqrt{d_X(x,x')^2 + d_Y(y,y')^2}$$

If there are geodesics on X and Y, we can define geodesics on $X \times Y$, and hence can define the geodesic flip, which can be shown to satisfy the formula

$$I_{(x,y)}(x',y') = (I_x(x'), I_y(y'))$$

In the case $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, this corresponds to the fact that the composition of a flip about a vertical line with a flip about a horizontal line is equivalent to rotation by π around the intersection of the two lines.

With the geodesic flip defined, we can then ask whether the product space $X \times Y$ is symmetric, and it turns out that if X and Y are both symmetric spaces, so is their direct product $X \times Y$. In this manner we can obtain many higher-dimensional examples, and so if we were to attempt to classify such spaces, we would want to focus on those which are irreducible in that they cannot be decomposed as the direct product of two lower-dimensional spaces, since the other examples will be built from these.

The direct product provides a common means by which we decompose objects of interest into simpler examples in order to gain a complete understanding. We find many examples of this in linear algebra, in which context the phrase *direct sum* is also sometimes used. Any finite-dimensional vector space can be written as the direct product of n copies of $\mathbb{R}$; this is just the statement that any finite-dimensional vector space has a basis. A more sophisticated application of this process is the decomposition of a linear transformation in terms of its action upon its eigenspaces, so that a symmetric matrix can be written as the direct product of one-dimensional transformations, while for a general matrix, we have the Jordan normal form.

This process is also used in the classification of finitely generated abelian groups, where we decompose the group of interest into a direct sum of copies of $\mathbb{Z}$ and cyclic groups whose order is a power of a prime, so that no further decomposition is possible. Thus the natural

counterpart to the study of how a particular sort of mathematical structure can be decomposed is the study of what instances of that structure are, in some appropriate sense, *irreducible*.