DMITRY DOLGOPYAT

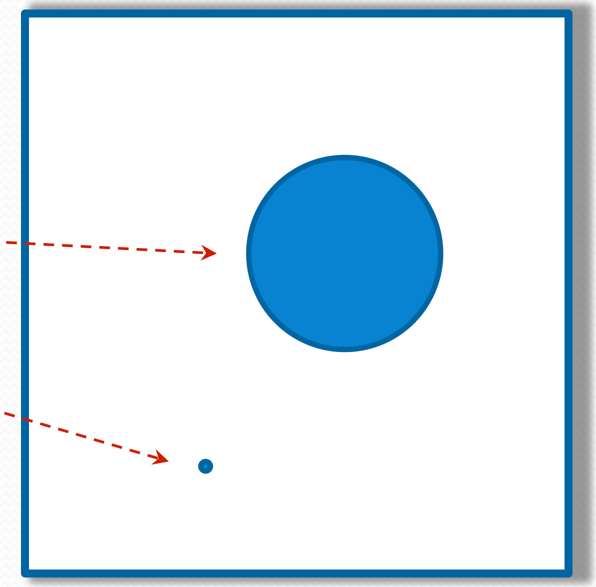
Works on physical models with moving particles

Brownian Motion

Model: two particles in a box

Heavy disk of mass M»1

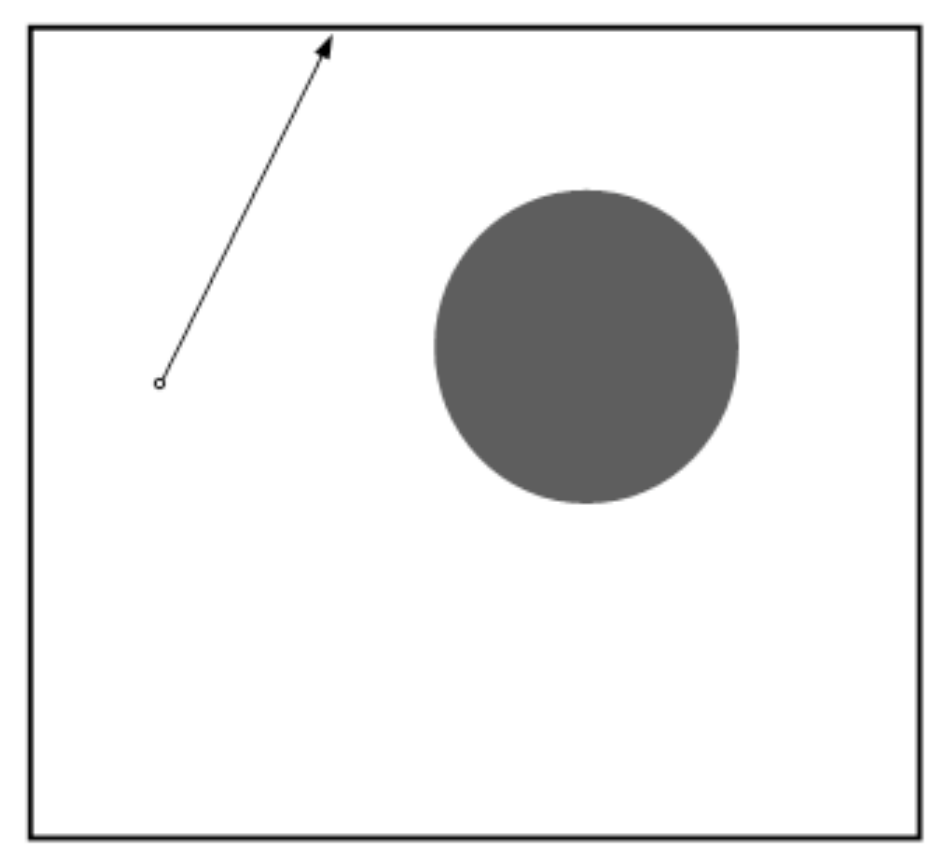
Light disk of mass m=1



Particles collide elastically

(preserving total kinetic energy and total momentum)

Particles bounce off the walls like billiard balls



Animated motion of the light particle

Questions:

Initial state (*Q,V*) of the heavy disk is fixed (preset)

Initial state (*q,v*) of the light disk is randomly chosen

Does *Q(t)* or *V(t)* behave as a Brownian trajectory?

Describe the motion of the heavy disk in the limit

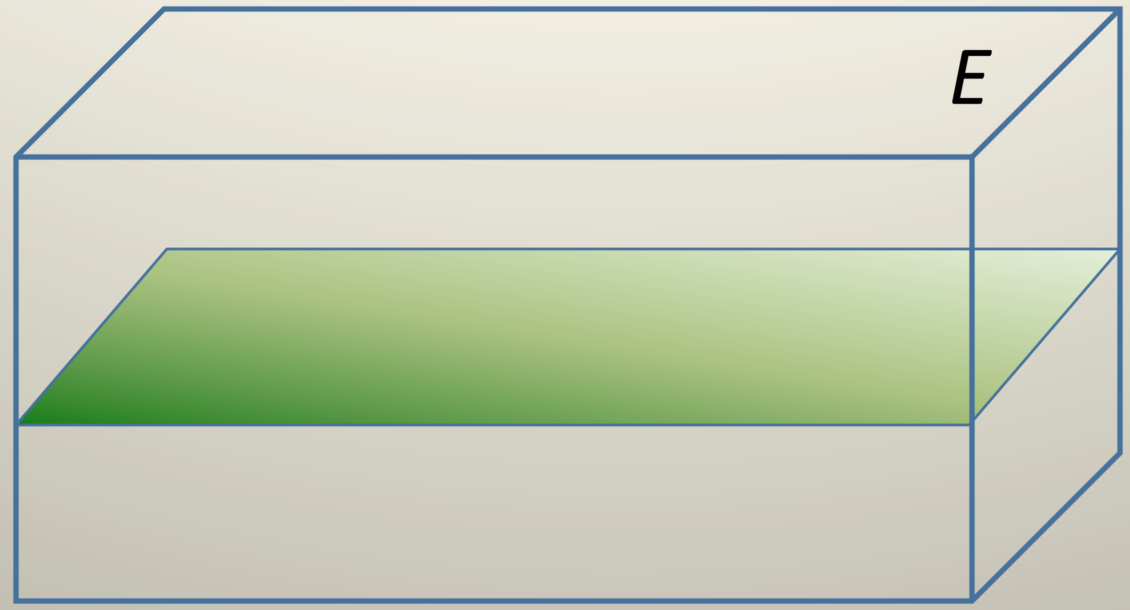
$$M \to \infty$$

Traditional approaches:

Full phase space **Ω** is 8-dimensional
(energy `surface' ***E*** is 7-dimensional)
Invariant measure **μ** is Liouville measure on **Ω**

Is the system hyperbolic? Open question…
Is the system ergodic? Open question…

The measure **μ**, hyperbolicity, and ergodicity are all irrelevant to the goals of the project.



Initial distribution is on a 3-D surface in the 7-D phase space *E*.

Its images remain close to itself.

Key feature is strong *partial hyperbolicity*:

- **one** strongly expanding direction (u)
- **one** strongly contracting direction (s)

In other directions evolution is very slow.

Key idea: Study images of measures supported on 1-D unstable curves.
Curve + Measure is called a **Standard Pair**

Conclusions:

1. The velocity *V(t)* of the heavy disk weakly converges to a Wiener-type process described by certain stochastic differential equations.

2. The position *Q(t)* of the heavy disk weakly converges to the integral of the above Wiener-type process.

Thus the Brownian motion (=Wiener process) characterizes the velocity *V(t)* of the heavy disk

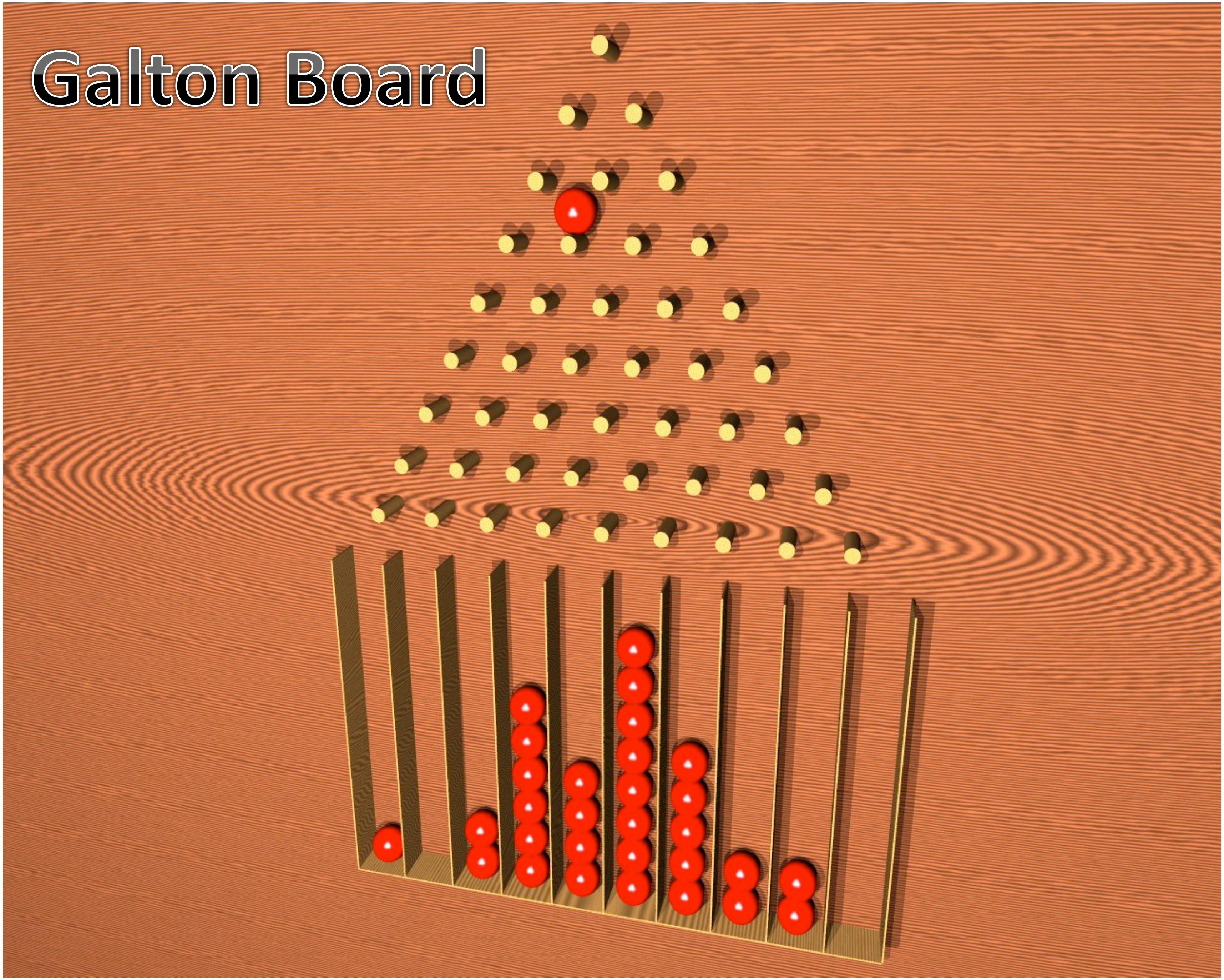
The position of the heavy disk (which was originally observed by R. Brown) may be now called "Brownian Brownian motion"

This work is published in

Brownian Brownian Motion – I

Memoirs AMS, Vol. 198, No 927 (2009)
(it is 193 pages long)

Galton Board



Galton board

An upright board with a periodic array of fixed pegs on which balls are rolling down bouncing off the pegs

Introduced by Sir Francis Galton (1822-1911)

Resembles a modern pinball machine

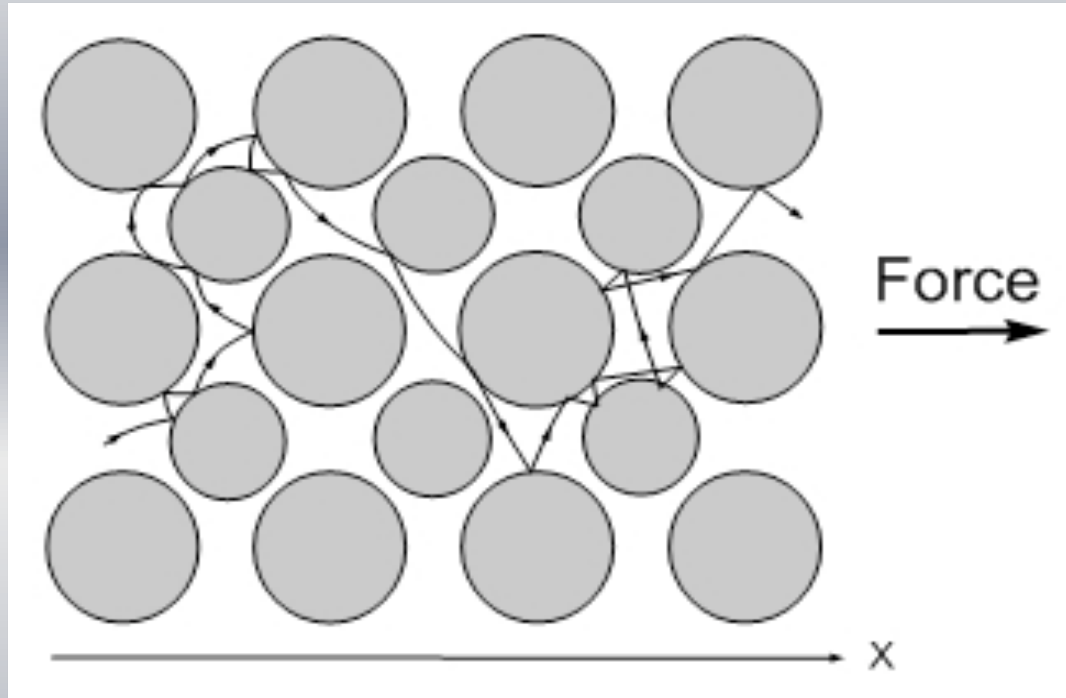
Mathematical model:

Periodic array of fixed convex domains (scatterers) on an infinite plane

Particle moves between the scatterers subject to a constant force (external field)

Particle bounces off the scatterers as a billiard ball

This model is also known as **Lorentz gas**:



Electrons move between ions in a metal subject to a voltage difference (electric field)

Ions are fixed, make a periodic (crystal) structure

Questions:

Describe asymptotic behavior of the position and velocity of the particle as time $t \to \infty$.

Physicists conjectured (based on heuristic and empirical studies):

Position $X(t) \sim t^{2/3}$ Velocity $V(t) \sim t^{1/3}$

No mathematical results until now…

Difficulties:

- Particle accelerates as it moves away
- Phase space is not compact, invariant measure is infinite
- Initial distribution is concentrated in a compact domain (say, where $0 \le X(0) \le 1$)
- Images of the initial measure escape to infinity
- Dynamics inhomogeneous in time and space

Results obtained:

Average position $X(t)$ grows as $t^{2/3}$

Average velocity $V(t)$ grows as $t^{1/3}$

Rescaled position $t^{-2/3}X(t)$ has a limit distribution

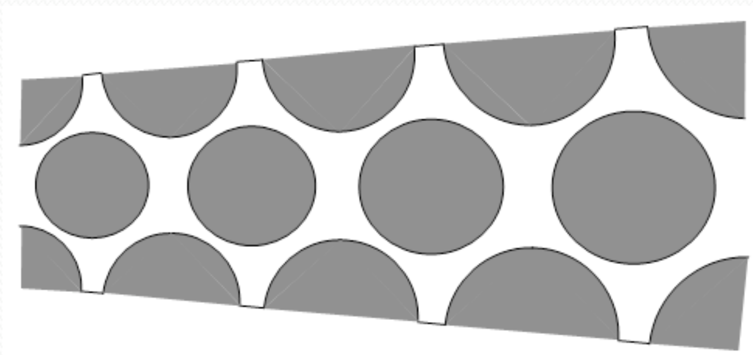
Rescaled velocity $t^{-1/3}V(t)$ has a limit distribution

Rescaled position converges to an Itô diffusion process satisfying certain Stochastic Diff. Eqs.

$X(t)$ is recurrent: the particle returns to its initial value $X(0)$ infinitely many times.

Self-similar billiards

Infinite channel of similar billiard cells proposed by F. Barra, T. Gilbert, and M. Romo:



Size grows exponentially (by a factor $\lambda>1$)

Particle moves in the channel and bounces off the walls.

Question: does $X(t)/t$ converge to a limit as $t \to \infty$?

Physicists conjectured: yes, it does…

Difficulties:

- Phase space is not compact, invariant measure is infinite
- Initial distribution is concentrated in one cell
- Images of the initial measure escape to infinity
- Dynamics inhomogeneous in time and space

Results obtained:

$X(t)/t$ does **NOT** have a limit distribution…

$X(t_n)/t_n$ has a limit distribution provided

$$\{ \log(t_n)/\log(\lambda) \} \to \rho \qquad 0 \le \rho < 1$$

(here $\lambda > 1$ is the factor of expansion of cells)

Thus, limit distributions of $X(t)/t$ change cyclically as $t \to \infty$

This fact was later confirmed in a computer experiment by physicists…

Lorentz gas with a thermostat

Electrons move subject to a force:

$$d\mathbf{q}/dt = \mathbf{p} \qquad d\mathbf{p}/dt = \mathbf{E}$$

q denotes the position

p denotes the momentum (velocity)

E denotes the electric field (external force)

Lorentz gas with a thermostat

Electrons move subject to a force:

$$d\mathbf{q}/dt = \mathbf{p} \qquad\qquad d\mathbf{p}/dt = \mathbf{E}$$

Lorentz gas with a thermostat

Electrons move subject to a force:

$$d\mathbf{q}/dt = \mathbf{p} \qquad d\mathbf{p}/dt = \mathbf{E} - \langle \mathbf{E},\mathbf{p} \rangle \mathbf{p}$$

Gaussian thermostat

Lorentz gas with a thermostat

Electrons move subject to a force:

$$d\mathbf{q}/dt = \mathbf{p} \qquad d\mathbf{p}/dt = \mathbf{E} - \langle \mathbf{E},\mathbf{p}\rangle \mathbf{p}$$

Now $\langle \mathbf{p},\mathbf{p}\rangle=1$ at all times.

In other words, the kinetic energy is kept constant.

The extra term prevents the electrons from speeding (*heating up*) or slowing down (*cooling down*).

It keeps the *temperature* fixed. Hence its name:

thermostat.

General features

Constant speed => Dynamics spatially periodic
=> Phase space is compact
=> Finite invariant measure

Unique Sinai-Ruelle-Bowen (SRB) measure μ
(singular, but abs. cont. on unstable fibers)

Electrical current $J=\mu(\mathbf{p})$ is average momentum of moving electrons.

EXISTING RESULTS:

Assume that horizon in the Lorentz gas is finite (free flights of the electrons are bounded)

Then the electrical current satisfies

$$\mathbf{J} = \mathbf{C}\mathbf{E} + o(|\mathbf{E}|) \qquad \text{(Ohm's law)}$$

Electrical conductivity **C** satisfies

$$\mathbf{C} = \tfrac{1}{2}\mathbf{D} \qquad \text{(Einstein relation)}$$

D is the diffusion matrix:

$$\mathbf{q}(t)/\sqrt{t} \Rightarrow N(0,\mathbf{D})$$

(N.Chernov, G.Eyink, J.Lebowitz, Ya.Sinai 1993)

NEW RESULTS:

Assume that horizon in the Lorentz gas is infinite (there are infinite corridors between ions)

Then the electrical current satisfies

$$\mathbf{J} = \mathbf{CE}\ |\log(|\mathbf{E}|)| + O(|\mathbf{E}|)$$

Electrical super-conductivity **C** satisfies

$$\mathbf{C} = \tfrac{1}{2}\mathbf{D} \qquad \text{(analogue of Einstein relation)}$$

D is the super-diffusion matrix

$$\mathbf{q}(t)/\sqrt{t\log(t)} \Rightarrow N(0,\mathbf{D})$$

(N.Chernov & D. Dolgopyat 2009)