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THE WORK OF DMITRY DOLGOPYAT ON PHYSICAL MODELS WITH MOVING PARTICLES

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ABSTRACT. D. Dolgopyat is the winner of the second Brin Prize in Dynamical Systems (2009). This article overviews his remarkable achievements in a nontechnical manner. It complements two other surveys of Dolgopyat's work written by Y. Pesin and C. Liverani and published in this issue. This survey covers Dolgopyat's work on various physical models, including the Lorentz gas, Galton board, and some systems of hard disks.

1. INTRODUCTION

In their surveys, Y. Pesin and C. Liverani have reflected upon the contribution made by Dmitry Dolgopyat to the theory of dynamical systems. They described those as fundamental and far-reaching, capable of affecting the future development of this field. Dolgopyat's profound ideas and innovative techniques have indeed made a great impact on the modern theory of chaos and hyperbolic dynamics.

But Dolgopyat's scholarly interests are not limited to fundamental mathematical theories. He also takes pleasure in solving applied problems. Those involve physical models, in which realistic particles (molecules and electrons) move in various reservoirs, and equations of motion are set according to the laws of classical mechanics or electromagnetism. Along these lines, Dolgopyat produced a series of remarkable results. In particular, he found answers to several notoriously difficult questions in mathematical physics and statistical mechanics.

Physical systems are often inconvenient and unsuitable for direct application of conventional theories: the dynamics may have ugly singularities, the phase space may not be compact (and may have infinite Lebesgue measure), hyperbolicity may be weak (nonuniform, partial, or coexisting with elliptic islands), natural invariant measures may be infinite, etc. Then standard methods of ergodic theory and hyperbolic dynamics fail to work or require substantial adaptation. Often one has to develop new approaches and invent clever

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tricks to tackle various difficulties. This is where Dolgopyat's talent is particularly strong.

Here I will overview Dolgopyat's work on physical models. My presentation will be nontechnical and will focus on the innovative features of this work.

2. BROWNIAN MOTION IN A SYSTEM OF TWO DISKS

Consider a seemingly simple system of two particles: a heavy disk of radius R and mass M; and a small light disk of radius $r < R$ and mass $m \ll M$. These particles move freely in a two-dimensional container and collide elastically with each other and with the walls. The walls are made of molecules, *i.e.,* small disks that are fixed (unmovable).

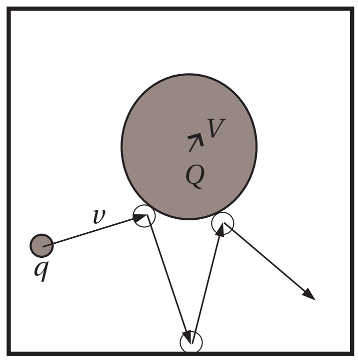

FIGURE 1. Two disks in a box.

Let $Q(t)$ denote the center and $V(t)$ the velocity of the heavy disk at time t. Similarly, let $q(t)$ denote the position of the light particle and $v(t)$ its velocity. Due to the elastic character of interaction between the disks, the total kinetic energy $E = \frac{1}{2}(M\|V\|^2 + m\|v\|^2)$ is conserved. The surface of constant energy E in the phase space of this system is a 7-dimensional compact manifold $\mathcal{M}$ with boundary. The dynamics on $\mathcal{M}$ is in fact a billiard flow Φ^t.

This is a Hamiltonian system, and it preserves the Liouville measure on $\mathcal{M}$ (which has a uniform density). Systems of hard disks in closed containers have been shown to be completely hyperbolic and ergodic under various conditions [10, 33], but those results do not cover this particular model. It remains unknown whether this model is hyperbolic or ergodic, and such questions seem to be beyond our abilities.

From a physicist's point of view, the main feature of this system is not hyperbolicity or ergodicity but the character of motion. Let us set, for convenience, $m = 1$ and $E = 1/2$. The heavy disk moves slowly; in fact, $\|V\| = \mathcal{O}(1/\sqrt{M})$. Moreover, at each collision with the small disk, the velocity $V(t)$ changes very little. In fact, its increment is $\|\Delta V\| = \mathcal{O}(1/M) \ll \|V\|$. This indicates that the velocity $V(t)$ behaves as a Brownian motion, in the limit $M \to \infty$. Then the position $Q(t)$ should behave as the integral of the Brownian motion (contrary to a common belief). The latter is also a stochastic process described by certain stochastic differential equations, but it is smoother than the classical Brownian motion (*i.e.,* Wiener process).

So the goal here is to prove (mathematically) that the trajectories of the heavy disk converge to a certain stochastic process and describe the latter by a system of stochastic differential equations. The tools required for the proof include methods of hyperbolic dynamics, averaging theory, probabilistic moment estimates, and elements of stochastic differential equations.

The small light disk moves at a higher speed $\|v\| = \mathcal{O}(1)$, and its velocity changes by $\mathcal{O}(1)$ at every collision. So the light disk behaves like a billiard ball that chaotically moves around and bombards the heavy disk from all sides. Thus the system has fast variables (q and v) and slow variables (Q and V). It is natural to fix the initial state (Q_0, V_0) of the heavy disk and select the state (q_0, v_0) of the light disk randomly anywhere in the available part of the phase space.

Thus our initial distribution μ_0 is supported on a 3-dimensional subspace $\Sigma_0 \subset \mathcal{M}$ (defined by $Q = Q_0$ and $V = V_0$), and on that subspace it is smooth (for simplicity, it can be chosen uniform). This is a probability measure that does not stay invariant, its image μ_t at time t is supported on a 3D surface Σ_t that changes with t, as the state of the heavy disk changes (if slowly). But it is exactly the measure μ_t that describes the distribution of trajectories $(Q(t), V(t))$ of the heavy disk.

One can visualize the support Σ_t of the measure μ_t as a 3D manifold roughly parallel to Σ_0 whose motion in the directions transverse to Σ_0 is very slow (as the heavy particle slowly changes its state). Within Σ_t there is one strongly expanding direction and one strongly contracting direction, because the light particle moves nearly as a billiard ball whose dynamics is known to be strongly hyperbolic. Roughly speaking, Σ_t gets stretched in the unstable direction, contracted in the stable direction (and cut by singularities). For large t, it appears as a collection of thin pieces that are long in the unstable direction and short (and getting ever shorter) in the stable direction.

This leads to the idea of foliating Σ_t by unstable curves and studying the evolution of each curve separately. The induced distribution on unstable curves will be smooth, approaching the Sinai–Ruelle–Bowen (SRB) distribution.

Dolgopyat proposed to study unstable curves with probability density on them. He called a curve with a density on it a *standard pair*. He proved that the images of a standard pair quickly become equidistributed in the available part of the phase space, and such properties as the decay of correlations and Central Limit Theorem hold for standard pairs just like they do for invariant measures.

Standard pairs happen to be a highly efficient and versatile tool in the studies of physical models. They can describe the evolution of a single curve as well as that of smooth measures supported on higher-dimensional manifolds (since the latter can be foliated by smooth unstable curves). Dolgopyat also designed a flexible method for controlling the evolution of standard pairs—the coupling techniques culminating in (now well-known) Coupling Lemma. All these tools were developed and honed during the work on this 'simple' model of two disks and latter employed in other studies [13, 15, 16, 17, 18].

In the end, it was proved indeed that the velocity process $V(t)$, after proper rescaling, converges to a Brownian motion-type process, and the position $Q(t)$ converges to its integral. See precise statements in [12]. In the most interesting case $V_0 = 0$ (the heavy disk is initially at rest) and the limiting processes satisfy the stochastic differential equations

$$dQ(\tau) = \tilde{V}\, d\tau \qquad \text{and} \qquad d\tilde{V}(\tau) = \sigma_Q\, dw(\tau),$$

where $\tau = tM^{-2/3}$ is the rescaled time, $\tilde{V}(\tau) = M^{2/3}V(t)$ the rescaled velocity, $w(\tau)$ is a standard 2D Wiener process, and σ_Q is the covariance matrix for the billiard system where the heavy disk with a fixed center Q plays the role of an obstacle.

This 'simple' system of two particles required a major effort. The work started in 2002 (when the model was proposed by Y. Sinai) and in its final form it was published in 2009. It became a 193-page book that appeared as a separate issue of Memoirs of the AMS.

3. GALTON BOARD

The Galton board [25, Chapter V] introduced by Sir Francis Galton (1822–1911), also known as quincunx or bean machine, is one of the simplest mechanical devices exhibiting stochastic behavior. It consists of an upright (or inclined) wooden board with rows of pegs. A ball thrown into the Galton board rolls under gravitation and bounces off the pegs on its way down. If many balls are thrown into the quincunx, then one can observe a normal distribution of balls coming to rest on the machine floor. This device can be found on display in the exhibit "Mathematica: a world of numbers... and beyond" by Charles and Ray Eames that is installed in the Boston Museum of Science, the New York Hall of Science and the Science and Technology Museum of Atlanta.

In mathematical studies, the board is $\mathbb{R}^2$ with a periodic array of round scatterers. The ball moves under a constant external field. It is common to impose a 'finite horizon' condition, *i.e.,* the ball cannot move in any direction indefinitely without hitting a scatterer.

Incidentally, this model is identical to a periodic Lorentz gas [27] that models the transport of electrons in metals. Without external fields, the periodic Lorentz gas reduces to a billiard system on its fundamental domain (a torus minus scatterers). This is a dispersing billiard (Sinai billiard); it preserves a Liouville (equilibrium) measure and has strong ergodic and statistical properties. In particular, it exhibits diffusive behavior, see [8, 9, 11, 34, 36].

On a Galton board, the ball moves under the gravitational field $\mathbf{g} = (g, 0)$ where we chose the coordinates in such a way that the x axis is aligned with the field. Let $\mathbf{q} = (x, y)$ denote the position and $\mathbf{v}$ the velocity of the ball. The

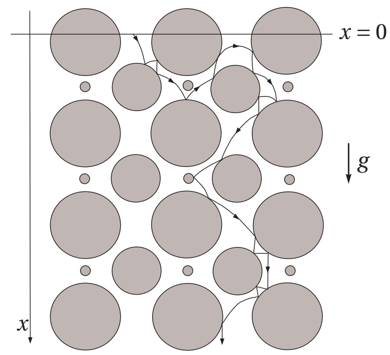

FIGURE 2. Galton board.

ball has unit mass, so the equations of motion are

$$d\mathbf{q}/dt = \mathbf{v}, \qquad d\mathbf{v}/dt = \mathbf{g}, \tag{3.1}$$

which preserve the total energy

$$E = \tfrac{1}{2} v^2 - gx, \tag{3.2}$$

where $v = \|\mathbf{v}\|$ is the ball's speed. Accordingly, when the ball rolls down the board, $x \to \infty$ and $v \sim x^{1/2} \to \infty$, so the speed grows to infinity. Thus the system does not have a finite invariant measure, and its energy surface is an unbounded 3D manifold, $\mathcal{M} = \{E = \text{const}\}$, with infinite Lebesgue measure. These features made mathematical studies prohibitively difficult.

Physicists were interested in this model since the 1970s. Numerous heuristic and numerical studies [26, 28, 29, 30] have indicated that the x-coordinate of the ball typically grows as $t^{2/3}$ and, respectively, its speed is $v \sim t^{1/3}$. But mathematically rigorous results were conspicuously lacking, until Dolgopyat's recent work [13].

It is natural to select the initial position of the particle randomly in a certain compact domain and choose the direction of its velocity uniformly on the unit circle. Then one gets an initial probability distribution μ_0 that has a compact 3D support in $\mathcal{M}$. It does not stay invariant as the particle travels in $\mathbb{R}^2$ and drifts in the direction of the field. So its image μ_t at time t will slowly change and spread over $\mathcal{M}$.

Dolgopyat used standard pairs again to study the evolution of μ_t. The support of μ_0 is foliated by unstable curves with induced densities, and then one can follow their images. But this system is quite different (and in many ways more difficult) than the two-disk model of the previous section; see below.

As the ball rolls down the board, its speed increases, and respectively it becomes less affected by the field. As a result, its trajectory between collisions with the obstacles becomes straighter, and its overall motion gets closer to that of a billiard ball without external field. Observe that the dynamics is inhomogeneous in space and time, its features gradually change.

More precisely, when the speed of the ball increases, the scattering effect produced by collisions gets stronger and the drift in the direction of the field gets weaker. Due to the stronger scattering effect, the ball bounces back more sharply and makes longer trips in the direction opposite to the field (where its speed decreases). This phenomenon is similar to Fermi, or diffusive shock acceleration [24, 37]. It ultimately leads to the recurrent character of the dynamics: the ball travels all the way back from time to time. More precisely,

$$\liminf_{t\to\infty} x(t) \le x(0)$$

with probability one. Such recurrent behavior is counterintuitive: we all know that a ball thrown into a real Galton board always rolls down and ends up on the floor. But on the idealized board, rather paradoxically, the ball will almost surely bounce all the way back up! (Of course, our idealized model does not take into account non-elasticity or air resistance or friction.)

Aside from the recurrence phenomenon, it was confirmed that indeed $x \sim t^{2/3}$ and $v \sim t^{1/3}$. More precisely, it was proven [13] that there is a constant $c > 0$ such that $c\,t^{-1/3}\,v(t)$ converges, as $t \to \infty$, to a random variable with density

$$\frac{3z}{\Gamma(2/3)} \exp\left[-z^3\right], \qquad z \ge 0.$$

Accordingly, $2gc^2t^{-2/3}x(t)$ converges to a random variable with density

$$\frac{3}{2\Gamma(2/3)} \exp\left[-z^{3/2}\right], \qquad z \ge 0.$$

These mathematical results fully resolved notoriously difficult questions discussed in the physics community for almost 40 years.

4. SELF-SIMILAR BILLIARDS

Barra, Gilbert, and Romo [4] introduced an interesting model—a channel of self-similar billiard tables with curved walls and inner obstacles and connected by passages, see Figure 3. In this infinite channel, a billiard ball travels free (under no external forces). The cells of this channel are similar to each other, and their sizes grow exponentially. More precisely, each cell is an enlarged copy of the previous one, expanded by the same factor $\lambda > 1$. Under these conditions, the opening in each cell on the right (into the bigger neighbor) is larger than the opening on the left (into the smaller neighbor). Since the billiard ball tends to move chaotically and its natural distribution is uniform in space, it is more likely that the ball will escape to the right than to the left. One can regard this model as a nonsymmetric 'random walk', thus expecting a steady drift of the billiard ball from left to right.

Physicists [4] studied the resulting dynamics heuristically and numerically, and they conjectured that the x coordinate of the ball grows linearly in time, *i.e.*, $x(t)/t$ converges to a limit distribution (which describes an 'average current').

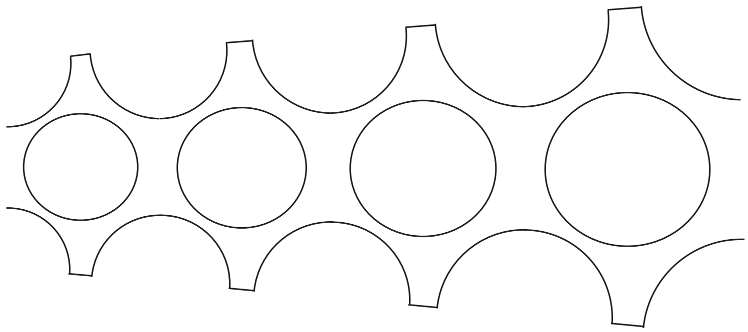

FIGURE 3. The Barra–Gilbert–Romo (BGR) channel.

It normally is physicists who discover new phenomena and laws based on plausible heuristic reasoning or numerical evidence. But in this particular instance it was mathematics that led to the truth. Dolgopyat's investigation revealed that $x(t)/t$ did not converge to a limit distribution, but rather fluctuated over a family of distributions.

More precisely, it was proved that if one chooses a sequence of time moments t_k such that the fractional part $\left\{\frac{\ln t_k}{\ln \lambda}\right\} \to \rho \in [0,1)$, then the ratio $q(t_k)/t_k$ has an asymptotic distribution (which, generally, depends on ρ). The proof clearly indicates that the limit distribution should depend on ρ.

Later more extensive computer experiments (see [3]) confirmed the dependence on ρ and further investigated properties of the system.

Mathematical studies of this model required further development of Dolgopyat's standard-pairs techniques. In this case the space and time inhomogeneity of the dynamics was even more dramatic than in the Galton board model. Basically, due to the exponential growth of the cells, the moving ball spends most of the time in the last few cells, hence averaging over cells does not work.

To handle this situation, Dolgopyat proposed to use a special discrete time in which only passages of the ball to one of the larger cells where it has not yet been were recorded. In this discrete time, the dynamics was hyperbolic but it failed to be one-to-one. This called for an adaptation of some basic definitions, such as those of Lyapunov exponents and unstable manifolds.

Such systems appeared even more naturally in later studies [14]. There we study a modification of the Galton board in which the energy of the falling ball was adjusted (reset) at every collision, to prevent the ball from unrestricted acceleration. In that case, at every collision the past energy is forgotten, so the system loses its memory. For this reason the past trajectory cannot be reconstructed uniquely. As it turns out, some phase states have multiple preimages while others have none. Nevertheless the dynamics have a clearly pronounced hyperbolicity—a strong expansion in one direction and a strong contraction in another.

The lack of invertibility, however, forced us to revise the very basic concepts of hyperbolic dynamics such as unstable manifolds, Lyapunov exponents, and SRB measures. We had to redefine and adjust some of these tools in such a way that they would apply to noninvertible hyperbolic maps. Similar work was done recently by Baladi and Gouëzel [1, 2].

5. LORENTZ GAS WITH A THERMOSTAT

If we interpret the external field in the Galton board as an *electric*, rather than gravitational, field, then we obtain a periodic Lorentz gas [27] that models the transport of electrons in metals.

A constant electric force should drive electrons and create a current. An electric current is characterized by a steady drift of electrons in the direction of the applied field. That is, the electron's position $x(t)$ should, on average, grow linearly in t, *i.e.,* we should have $x(t) \sim Jt$, where J is the numerical value of the current (in amperes).

The Lorentz gas model does not feature these properties, though, for two reasons. First, the electron's speed is unrestricted and tends to grow to infinity. Second, their drift is not linear in t, but rather $\sim t^{2/3}$, according to our description of the Galton board. These seemingly contradicting features (high speed and slow drift) were discussed earlier.

To keep the speed of the moving particle fixed and to make its drift proportional to t, Moran and Hoover [28] modified equations (3.1) as follows:

$$d\mathbf{q}/dt = \mathbf{v}, \qquad d\mathbf{v}/dt = \mathbf{E} - \zeta\mathbf{v}, \tag{5.1}$$

where $\zeta = \langle \mathbf{E}, \mathbf{v}\rangle / \|\mathbf{v}\|^2$. (We denote here the field by $\mathbf{E}$ because it is regarded as an electric field.) The friction term $\zeta\mathbf{v}$ is called the *Gaussian thermostat*, it ensures that $\|\mathbf{v}\|$ is constant in time. This makes the kinetic energy (*i.e.,* the temperature) constant, thus the term 'thermostat' – a device that prevents the system from heating up or cooling down.

The resulting dynamics is homogeneous in space and time—it follows the same rules in every cell of the Lorentz gas. So it can be projected onto a fundamental domain $\mathscr{D}$ (with periodic boundary conditions) and we obtain a system with compact 3D phase space $\Omega = \mathscr{D} \times \mathbb{S}^1$. It has a natural finite invariant measure.

For these reasons the above system is simpler than all that were described in the previous sections. No wonder it has been fully investigated in the pre-Dolgopyat era: it was proved in 1993 [19, 20] that the dynamics is hyperbolic and preserves a unique Sinai–Ruelle–Bowen (SRB) measure. That measure is mixing and enjoys an exponential decay of correlations. The average drift is linear, *i.e.,* $\langle \mathbf{q}(t)\rangle = \mathbf{J}t$, where the current $\mathbf{J}$ is proportional to the voltage difference:

$$\mathbf{J} = \mathbf{K}\mathbf{E} + o(\|\mathbf{E}\|), \tag{5.2}$$

where $\mathbf{K}$ is a constant known as electric conductivity; it is given by

$$\mathbf{K} = \tfrac{1}{2}\mathbf{D} \tag{5.3}$$

where $\mathbf{D}$ denotes the corresponding diffusion matrix for the unperturbed (field-free) Lorentz gas. In physics, (5.2) is known as *Ohm's law* and (5.3) as the *Einstein relation*.

All the above properties of the Lorentz gas have been proved under one assumption: *finite horizon*. This means that the fixed obstacles (molecules) are large enough (or dense enough) to block all possible directions in which the ball (electron) can move freely without collisions.

Systems with infinite horizon are much harder to investigate. There are infinite corridors between obstacles where the electron can fly arbitrarily far nonstop, ballistically. This leads to *superdiffusion* which has been investigated since the 1980s. In 1992, Bleher [5] published a partial proof of the long-standing conjecture that the ball spreads at a rate $\sqrt{t \log t}$ rather than $\sqrt{t}$. A complete mathematical proof for discrete time was published by Szasz and Varju in 2007; see [35]. They showed that the position $\mathbf{q}_n$ of the ball at the nth collision, rescaled as $\mathbf{q}_n/\sqrt{n \log n}$, converged to a normal distribution. Here one has to apply a Central Limit Theorem to a random variable with infinite second moment. To cope with this complication Szasz and Varju used nonclassical versions of the Central Limit Theorem.

Dolgopyat found a different approach: he cleverly truncated the unbounded random variable with infinite variance. His two-step truncation procedure led to a more elementary proof of the same fact, and to a proof of its real-time version: $\mathbf{q}(t)/\sqrt{t \log t}$ converges to a normal distribution as well. The covariance matrix $\mathbf{D}_s$ of the latter is called *superdiffusion matrix*.

When an electrical field is present in a Lorentz gas with infinite horizon, corridors not only cause superdiffusion but also lead to *superconductivity*. The current becomes abnormal and electrons travel faster than they do in the conventional case. More precisely, it was proved in the next work of Dolgopyat's [16] that

$$\mathbf{J} = \tfrac{1}{2}\left|\log(\|\mathbf{E}\|)\right| \mathbf{D}_s\mathbf{E} + o(\|\mathbf{E}\|), \tag{5.4}$$

where $\mathbf{D}_s$ denotes the superdiffusion matrix. Comparing this with (5.2)–(5.3), one can see that in the infinite horizon case Ohm's law fails, but the Einstein relation, suitably interpreted, holds.

The above mathematical theorem may be related to the physical phenomenon of superconductivity. At low temperatures (near absolute zero), ions tend to form an almost perfect crystal structure with long corridors in between resembling our infinite horizon model. Thus the electron tends to travel fast and one observes superconductivity. On the other hand, at normal temperatures, ions are somewhat agitated, their configuration is more randomized, which creates an effect of finite horizon, which slows the electron down and one observes a normal current.

In mathematical terms, this model has a compact phase space and a finite invariant SRB measure, but its key properties are highly nonuniform in the field $\mathbf{E}$ (and in many senses they deteriorate as $\mathbf{E} \to 0$). Many delicate estimates were necessary to control that nonuniformity.

6. CLASSICAL INFINITE LORENTZ GAS

Lastly we describe joint works by Dolgopyat, Szasz, and Varju on the classical periodic Lorentz gas without external fields. Due to its periodicity, it can be reduced to a billiard in a fundamental domain with periodic boundary conditions. That billiard has a finite invariant measure and was investigated by Sinai [34] and others [6, 7, 8] long ago.

But the properties of the infinite periodic gas, in the entire plane, remain largely unknown. That system has an infinite invariant measure and thus is much harder to study. In 1989, Simányi [32] proved that *if* it is recurrent, *then* it is ergodic. The recurrence means that if the ball hits a scatterer, then it will come back to that scatterer with probability one.

In 1998–1999, Conze [21] and Schmidt [31] proved recurrence independently and using different methods. Thus the system was proven to be ergodic. Its mixing properties largely remain obscure, and in fact there is no conventional definition of mixing for infinite-measure systems.

In 2008–2009, Dolgopyat, Szasz and Varju [35] conducted a detailed study of the infinite Lorentz gas and derived a series of quantitative results and estimates. First, let the R denote the return time of the bouncing ball to the initial cell (where it was at time 0). Then a logarithmic tail bound was proved in [22]:

$$\text{Prob}(R > n) \sim \frac{\beta}{\log(n)} \quad \text{as} \quad n \to \infty,$$

where $\beta = 2\pi\sqrt{\det \mathbf{D}}$ and $\mathbf{D}$ is the corresponding diffusion matrix.

Next, let k_n denote the number of collisions that occur in the initial cell during the period of the first n collisions. Then [22] for every $x > 0$

$$\text{Prob}\left(\frac{\beta k_n}{\log(n)} < x\right) \to 1 - e^{-x} \quad \text{as } n \to \infty,$$

which means a weak convergence to an exponential distribution with parameter 1. These both formulas are standard facts for classical 2D symmetric random walks. Thus the Lorentz gas behaves like a random walk, in a global sense.

Next, suppose we naturally label the cells in the Lorentz gas by pairs of integers (m,n) (in such a way that $(0,0)$ denotes the initial cell). Then for any $(m,n) \in \mathbb{Z}^2$ let $T(m,n)$ denote the time when the bouncing ball reaches the cell (m,n). It is proved in [22] that $T(m,n)$ grows like $m^2 + n^2$. More precisely, for any $0 < x < 1$

$$\text{Prob}\left(\frac{\log[T(m,n)]}{\log(m^2+n^2)} < x\right) \to x \quad \text{as } m^2 + n^2 \to \infty,$$

which means a weak convergence to a uniform distribution on the unit interval $(0,1)$. In addition, the distribution of the first landing point in the cell (m,n) converges to the natural invariant measure.

In the next paper [23] the same authors studied locally perturbed Lorentz gases. It was motivated by a old question raised by Sinai: if we change the position of a single scatterer in a periodic Lorentz gas, will the diffusive behavior of the ball be affected? In other words, will the position $\mathbf{q}(t)$ of the ball, rescaled by $\mathbf{q}/\sqrt{t}$, converge to a normal law?

It is a difficult question because after a local perturbation the gas is no longer periodic and cannot be reduced to a compact billiard. It still preserves an infinite measure and has to be studied as such.

Dolgopyat and his coauthors [23] answered the above question positively. They proved that if one modifies the location or shape of finitely many scatterers in a periodic Lorentz gas in $\mathbb{R}^2$, then $\mathbf{q}/\sqrt{t}$ still converges to the normal law $N(0,\mathbf{D})$, where $\mathbf{D}$ is the same diffusion matrix as for the unperturbed gas. They also proved that the trajectory of the ball, properly rescaled, converged to a Brownian motion.

7. FINAL REMARKS

It must be emphasized that all Dolgopyat's papers described in this article involved an enormous amount of work. Most of the physical models are technically complicated and require lengthy analytic calculations. While many proofs were based on the same general approach, none could use the results of the others directly, every model was in a way unique and each required working through all the stages of the proofs anew.

Not surprisingly, most of the above papers are long—40 or 50 journal pages—and this is after much effort was spent on optimizing the arguments and compressing the presentation. A notable exception is [12] which spans 193 pages. It is regarded as a first step in a long-term research program, which is why the numeral one appears in its title. So Dolgopyat's quest for discoveries in mathematical physics has not reached its peak yet—we will hear more from him in the future.

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