The Second Brin's Prize in Dynamical Systems

On the Work of Dolgopyat on Partial and Nonuniform Hyperbolicity

Ya. Pesin

Stable Ergodicity

Let $f : M \to M$ be a C^r diffeomorphism, $r \geq 1$ of a compact smooth connected Riemannian manifold M preserving a Borel probability measure μ. It is called *stably ergodic* if there exists a neighborhood $\mathcal{U} \subset \mathsf{Diff}^k(M, \mu)$ (the space of C^k diffeomorphisms, $k \leq r$, preserving the measure μ) of f such that any C^r diffeomorphism $g \in \mathcal{U}$ is ergodic.

Similarly, one can define the notions of systems being *stably mixing*, *stably Kolmogorov* and *stably Bernoulli*.

The Conservative Case

f is a partially hyperbolic diffeomorphism preserving a *smooth* measure μ. f possesses an invariant decomposition of the tangent bundle:

$$TM = E^s \oplus E^c \oplus E^u, \;\; df E^{s,c,u}(x) = E^{s,c,u}(f(x))$$

and uniform expansion and contraction rates along these subspaces:

$$\lambda_1 < \nu_1 \leq \nu_2 < \lambda_2, \;\; \lambda_1 < 1 < \lambda_2.$$

The distributions E^s and E^u are integrable to invariant transversal continuous foliations with smooth leaves W^s and W^u. These foliations possess absolute continuity property, i.e., the conditional measures μ^s and μ^u generated by μ on local stable and unstable manifolds are equivalent to leaf volumes m^s and m^u.

The central distribution may or may not be integrable and even if it does the central foliation my not be absolutely continuous.

Two points x and y are *accessible* if there is a path connecting them and consisting of pieces of stable and unstable manifolds. f is *accessible* if any two points are accessible and is *essentially accessible* if the partition by accessibility classes is trivial. f is *center-bunched* if $\lambda_1 < \nu_1 \nu_2^{-1}$ and $\lambda_2 > \nu_2 \nu_1^{-1}$.

Theorem (Burns-Wilkinson). Assume that f is C^2, essentially accessible and center-bunched. Then f is ergodic. If in addition, f is stably essentially accessible then it is stably ergodic in $\mathrm{Diff}^1(M, \mu)$.

This result provides a partial solution of the *Pugh-Shub stable ergodicity conjecture* for partially hyperbolic diffeomorphisms. When the center direction is one-dimensional the center-bunched condition can be dropped leading to a complete solution of the conjecture: stable essential accessibility implies stable ergodicity (Burns-Wilkinson, Hertz-Hertz-Ures).

Accessibility

The first result on genericity of accessibility was obtained by Dolgopyat and Wilkinson.

Theorem. Let $f \in \mathrm{Diff}^q(M)$ ($f \in \mathrm{Diff}^q(M, \mu)$), $q \geq 1$, be partially hyperbolic. Then for every neighborhood $U \subset \mathrm{Diff}^1(M)$ ($U \subset \mathrm{Diff}^1(M, \mu)$) of f there is a C^q diffeomorphism $g \in U$ which is stably accessible.

The proof uses Brin's quadrilateral argument. Given a point $p \in M$, let $[z_0, z_1, z_2, z_3, z_4]$ be a 4-legged path originating at $z_0 = p$. Connecting z_{i-1} with z_i by a geodesic γ_i lying in the corresponding stable or unstable manifold, we obtain the curve $\Gamma_p = \cup_{1 \leq i \leq 4} \gamma_i$. We parameterize it by $t \in [0, 1]$ with $\Gamma_p(0) = p$. If the distribution $E^s \oplus E^u$ were integrable (and hence, the accessibility property would fail) the endpoint $z_4 = \Gamma_p(1)$ would lie on the leaf of the corresponding foliation passing through p.

Therefore, one can hope to achieve accessibility by arranging a 4-legged path in such a way that $\Gamma_p(1) \in W^c(p)$ and $\Gamma_p(1) \neq p$. In this case the path Γ_p can be homotoped through 4-legged paths originating at p to the trivial path so that the endpoints stay in $W^c(p)$ during the homotopy and form a continuous curve. Such a situation is usually persistent under small perturbations of f and hence leads to stable accessibility.

In the special case of 1-dimensional center bundle, Didier has shown that accessibility is an open dense property in the space of diffeomorphisms of class C^2.

Negative (positive) central exponents

A partially hyperbolic diffeomorphism f has negative (respectively, positive) central exponents if there is a set $A \subset M$ of positive ν-measure such that for every $x \in A$ and every $v \in E^c(x)$ the Lyapunov exponent $\chi(x, v) < 0$ (respectively, $\chi(x, v) > 0$).

Theorem (Burns-Dolgopyat-Pesin). Assume that f is C^2, essentially accessible and has negative (or positive) central exponents. Then f is stably ergodic in $\mathrm{Diff}^1(M, \mu)$.

The Dissipative Case

$f : M \to M$ is a C^2 diffeomorphism of a compact manifold M.

Λ is an *attractor* if it is compact invariant and there exists an open neighborhood U of Λ s.t. $\overline{f(U)} \subset U$ and $\Lambda = \bigcap_{n \geq 0} f^n(U)$. U is the *basin of attraction*.

Λ is a *partially hyperbolic attractor* if it is an attractor for f and $f|\Lambda$ is partially hyperbolic, i.e., the tangent bundle $T\Lambda$ admits an invariant splitting $T\Lambda = E^s \oplus E^c \oplus E^u$ into stable, center, and unstable subbundles.

E^u is integrable; Λ is the union of the global strongly unstable manifolds of its points, i.e., $W^u(x) \subset \Lambda$ for every $x \in \Lambda$.

A measure μ on Λ is called a u-*measure* if for a.e. $x \in \Lambda$ the conditional measure $\mu^u(x)$ generated by μ on $W^u(x)$ is equivalent to the leaf volume $m^u(x)$ on $W^u(x)$.

Problems

1. Existence of u-measures.

2. Relations between u-measures and SRB-measures; in particular, between the basins of u-measures and the basin of attraction.

3. (non)uniqueness of u-measures.

4. u-measures with negative central exponents; ergodic properties and examples. Uniqueness of u-measures with negative central exponents.

5. Stability of u-measures under small perturbations of the map.

Existence of u-measures

Starting with a measure κ in a neighborhood U of Λ, which is absolutely continuous w.r.t. the Riemannian volume m, consider its evolution,

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} f_*^i \kappa. \qquad (1)$$

Any limit measure μ is concentrated on Λ.

Theorem (Pesin-Sinai, Bonatti-Diaz-Viana). Any limit measure μ is a u-measure.

Fix $x \in \Lambda$ and consider a local unstable leaf $V^u(x)$ through x. We can view the leaf volume $m^u(x)$ on $V^u(x)$ as a measure on the whole of Λ. Consider its evolution

$$\nu_n = \frac{1}{n} \sum_{i=0}^{n-1} f_*^i m^u(x). \qquad (2)$$

Any limit measure ν is concentrated on Λ.

Theorem (Pesin-Sinai). Any limit measure of the sequence (2) is a u-measure.

The basin of the measure

Given an invariant measure μ on Λ, define its *basin* $B(\mu)$ as the set of points $x \in M$ for which the Birkhoff averages $S_n(\varphi)(x)$ converge to $\int_M \varphi \, d\mu$ as $n \to \infty$ for all continuous functions φ.

If Λ is a hyperbolic attractor then μ is an SRB measure iff its basin has positive measure.

Theorem (Bonatti-Diaz-Viana). Any measure with basin of positive volume is a u-measure.

While any partially hyperbolic attractor has a u-measure, measures with basins of positive volume need not exist (just consider the product of the identity map and a diffeomorphism with a hyperbolic attractor).

Theorem (Dolgopyat). If there is a unique u-measure for f in Λ, then its basin has full volume in the topological basin of Λ.

u-measures with negative central exponents

μ is a u-measure for f. We say that f has *negative central exponents* if there is $A \subset \Lambda$ with $\mu(A) > 0$ s.t. the Lyapunov exponents $\chi(x,v) < 0$ for any $x \in A$ and $v \in E^c(x)$.

Theorem (Burns-Dolgopyat-Pesin-Pollicott). Assume that: 1) there exists a u-measure μ for f with negative central exponents; 2) for every $x \in \Lambda$ the global unstable manifold $W^u(x)$ is dense in Λ. Then

(1) μ is the only u-measure for f and hence, the unique SRB measure;

(2) f has negative central exponents at μ-a.e. $x \in \Lambda$; (f, μ) is ergodic and indeed, is Bernoulli;

(3) the basin of μ has full volume in the topological basin of Λ.

Constructing negative central exponents

There are partially hyperbolic attractors for which *any* u-measure has zero central exponents (the product of an Anosov map and the identity map of any manifolds).

There are partially hyperbolic attractors which allow u-measures with negative central exponents but not every global manifold $W^u(x)$ is dense in the attractor (the product of an Anosov map and the map of the circle leaving north and south poles fixed).

Small perturbations of systems with zero central exponents.

(1) Shub and Wilkinson considered small perturbations F of the direct product $F_0 = f \times Id$, where f is a linear Anosov diffeo and the identity acts on the circle. They constructed F in such a way that it preserves volume, has negative central exponents on the whole of M and its central foliation is not absolutely continuous ("Fubini's nightmare").

(2) Ruelle extended this result by showing that for an open set of one-parameter families of (not necessarily volume preserving) maps F_ϵ through F_0, each map F_ϵ possesses a u-measure with negative central exponent.

(3) Dolgopyat showed that, in the class of skew products, negative central exponents appear for generic perturbations and that there is an open set of one-parameter families of skew products near $F_0 = f \times Id$ (f is an Anosov diffeomorphism and Id is the identity map of any manifold) where the central exponents are negative with respect to any u-measure.

(4) Dolgopyat also considered a one-parameter family f_ϵ where f_0 is the time-1 map of the geodesic flow on the unit tangent bundle over a negatively curved surface. It is shown that in the volume-preserving case, generically, either f_ϵ or f_ϵ^{-1} has negative central exponent for small ϵ and that there is an open set of non conservative families where the central exponent is negative for any u-measure.

(6) Barraveira and Bonatti proved that if all the Lyapunov exponents in the central directions are zero then by an arbitrary small perturbation one can obtain that their sum can be made negative on a set of positive measure.

Systems with zero central exponents subjected to rare kicks.

Given diffeomorphisms f and g, let $F_n = f^n \circ g$. Dolgopyat has shown that if f is either a T^1-extension of an Anosov diffeomorphism or the time-1 map of an Anosov flow and g is close to Id, then, for typical g and any sufficiently large n, either F_n or F_n^{-1} has negative central exponent with respect to any u-measure.

Stable ergodicity for dissipative systems

Any C^1 diffeomorphism g sufficiently close to f in the C^1 topology has a hyperbolic attractor Λ_g which lies in a small neighborhood of Λ_f.

Theorem (Burns-Dolgopyat-Pesin-Pollicott). Let f be a C^2 diffeo with a partially hyperbolic attractor Λ_f. Assume that 1) there is a u-measure μ for f with negative central exponents on a subset $A \subset \Lambda_f$ of positive measure; and 2) for every $x \in \Lambda_f$ the global strongly unstable manifold $W^u(x)$ is dense in Λ_f. Then any C^2 diffeomorphism g sufficiently close to f in the $C^{1+\alpha}$-topology (for some $\alpha > 0$) has negative central exponents on a set of positive measure with respect to a u-measure μ_g. This measure is the unique u-measure (and SRB measure) for g, $g|\Lambda_g$ is ergodic with respect to μ_g (indeed is Bernoulli), and the basin $B(\mu_g)$ has full volume in the topological basin of Λ_g.

Attractors with positive central exponents

Alves, Bonatti and Viana obtained an ergodicity result under the stronger assumption that there is a set of positive volume in a neighborhood of the attractor with positive central exponents.

Vasquez proved a stable ergodicity result.

Theorem. Let f be a C^2 diffeo with a partially hyperbolic attractor Λ_f. Assume that:

1) there is a **unique** u-measure μ for f with positive central exponents on a subset $A \subset \Lambda_f$ of **full** measure;

2) for every $x \in \Lambda_f$ the global strongly unstable manifold $W^u(x)$ is dense in Λ_f.

Then f is stably ergodic.

Presence of nonuniformly hyperbolic dynamical systems on any manifold

Theorem (Dolgopyat-Pesin). Given a compact smooth Riemannian manifold $K \neq S^1$ there exists a C^∞ diffeomorphism f of K such that

1. f preserves the Riemannian volume m;

2. f has nonzero Lyapunov exponents a.e. ;

3. f is a Bernoulli diffeomorphism.

Katok's Example

There exists an area-preserving C^∞ diffeo of the disk D^2 s.t.

(1) g has nonzero Lyapunov exponents a.e.

(2) g is uniformly hyperbolic outside a small neighborhood U of the *singularity set* $Q = \partial D^2 \cup \{p_1, p_2, p_3\}$, i.e., there exists $\lambda > 1$, s.t.

$$\|dg|E_g^s(x)\| \leq \frac{1}{\lambda}, \quad \|dg^{-1}|E_g^u(x)\| \leq \frac{1}{\lambda}.$$

(3) g has two invariant stable and unstable foliations, W_g^s, W_g^u of $D^2 \setminus Q$ with smooth leaves. The foliations are continuous and absolutely continuous.

Brin's Example

1. A is a volume preserving hyperbolic automorphism of the torus T^{n-3}.

2. $\tilde{T}^t$ is a the suspension flow over A with a constant roof function. The flow $\tilde{T}^t$ is Anosov but does not have the accessibility property. However, one can perturb the roof function s.t. the new flow T^t (which is still Anosov) does have the accessibility property.

The phase space Y^{n-2} of T^t is diffeomorphic to the product $T^{n-3} \times [0,1]$, where the tori $T^{n-3} \times 0$ and $T^{n-3} \times 1$ are identified by the action of A.

3. The skew product R on $D^2 \times Y^{n-2}$

$$R(x,y) = (g(x), T^{\alpha(x)}(y)),$$

where α is a non-negative function on D^2 which is equal to zero in the neighborhood U of the singularity set Q and is strictly positive otherwise.

Properties of the Map R

$\Gamma = Q \times Y^{n-2}$ is the *singularity set* for R, $\Omega = (D^2 \setminus U) \times Y^{n-2}$

1. R is uniformly partially hyperbolic on Ω:

$$T_z M = E_R^s(z) \oplus E_R^c(z) \oplus E_R^u(z), \quad z \in \Omega$$

and for some $\mu > 1$

$$\|dg|E_R^s(z)\| \leq \frac{1}{\mu}, \quad \|dg^{-1}|E_R^u(z)\| \leq \frac{1}{\mu}.$$

2. The distributions $E_R^s(z)$ and $E_R^u(z)$ generate two C^1 continuous foliations W_R^s and W_R^u on $M \setminus \Gamma$.

3. R has essential accessibility property.

4. $m\{x \in M : R^n(x) \in U\} = 0$.

The Embedding

There is a smooth embedding

$$\chi_1 : D^2 \times Y^{n-2} \to B^n$$

which is a diffeo except for the boundary $\partial D^2 \times Y^{n-2}$. There is a smooth embedding $\chi_2 : B^n \to M$ which is a diffeo except for the boundary ∂B^n. Since the map R is identity on the boundary $\partial D^2 \times Y^{n-2}$ the map $h = (\chi_1 \circ \chi_2) \circ R \circ (\chi_1 \circ \chi_2)^{-1}$ has the following properties:

1. h preserves the Riemannian volume;

2. h is a Bernoulli diffeo.;

3. h has only one zero Lyapunov exponent.

The Perturbation

Given $r > 0$ and $\varepsilon > 0$, there is a C^r diffeo $P : M \to M$ which preserves volume m and s.t.

(1) $d_{C^r}(P, R) \leq \varepsilon$ and P is gentle, i.e., P is concentrated outside the singularity set Ω;

(2) a.e. orbit of P is dense in M;

(3) for a.e. $z \in M$ there exists a decomposition

$$T_z M = E_P^s(z) \oplus E_P^c(z) \oplus E_P^u(z)$$

s.t. $\dim E_P^c(z) = 1$ and

$$\int_M \chi_P^c(z)\, dm < 0,$$

where $\chi_P^c(z) = \lim_{n \to \infty} \frac{1}{n} \log \|df^n | E_P^c(z)\|$ is the Lyapunov exponent at $z \in M$ in the central direction.

Choose a coordinate system $\{x, \xi\}$ s.t. $dm = \rho(x, \xi)dx\,d\xi$ and

$$E_T^c(y_0) = \frac{\partial}{\partial \xi_1}, \quad E_T^s(y_0) = \langle \frac{\partial}{\partial \xi_2}, \ldots, \frac{\partial}{\partial \xi_k} \rangle,$$

$$E_T^u(y_0) = \langle \frac{\partial}{\partial \xi_{k+1}}, \ldots, \frac{\partial}{\partial \xi_{n-2}} \rangle$$

for some k, $2 \leq k < n - 2$. Let $\psi(t)$ be a C^∞ function with compact support and $\tau = \frac{1}{\gamma^2}(\|x\|^2 + \|\xi\|^2)$. Define

$$\varphi(x, \xi) = (x, \xi_1 \cos{(\varepsilon\psi(\tau))} + \xi_2 \sin{(\varepsilon\psi(\tau))},$$

$$-\xi_1 \sin{(\varepsilon\psi(\tau))} + \xi_2 \cos{(\varepsilon\psi(\tau))}, \xi_3, \ldots, \xi_{n-2}).$$