JOURNAL OF MODERN DYNAMICS
VOLUME 4, NO. 2, 2010, 211–225

doi: 10.3934/jmd.2010.4.211

ON THE WORK AND VISION OF DMITRY DOLGOPYAT

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(Communicated by Anatole Katok)

ABSTRACT. We present some of the results and techniques due to Dolgopyat. The presentation avoids technicalities as much as possible while trying to focus on the basic ideas. We also try to present Dolgopyat's work in the context of a research program aimed at enlightening the relations between dynamical systems and nonequilibrium statistical mechanics.

1. INTRODUCTION

This paper is occasioned by the award of the second Michael Brin prize in Dynamical Systems to Dmitry Dolgopyat. I will try to explain not only some of the results for which the prize has been awarded but also the general relevance of Dolgopyat's work for the future development of the field of dynamical systems.

The modern field of dynamical systems finds its roots not only in the study of celestial mechanics and, most notably, in Henri Poincaré's recognition of the phenomenon of dynamical instability [44], but also in the work of Ludwig Boltzmann concerning the foundations of statistical mechanics. Indeed, the very concept of *ergodicity*, a cornerstone in the study of dynamical systems, is due to Boltzmann [7].

Yet, if one wants to obtain results relevant for statistical mechanics, it is necessary to reckon with systems having a large number of degrees of freedom. Up to now, apart from few cases, we are able to deal only with few degrees of freedom (low-dimensional dynamics).[1] In spite of this limitation the field of dynamical systems has been able to produce some results relevant to statistical mechanics[2] but their relevance is limited by the fact that they essentially

Received February 14, 2010, revised May 14, 2010.

2000 *Mathematics Subject Classification:* Primary: 37D50; Secondary: 82C70.

Key words and phrases: Brin prize, Dolgopyat,

[1]The only real exceptions are, to my knowledge, coupled-map lattices (see [34, 35] for the latest results and references). It is true that one can establish exponential decay of correlations for geodesic flows on manifolds of negative curvature in any dimension [40], but since nothing is known concerning the dependence of the rate of decay on the dimension, this knowledge is not really very useful in higher dimensions.

[2]A few examples are: the Hénon and Lorenz attractors [5, 51], the work on Lorentz gas started by [9] (see [24] for latest results and references), linear response theory [46], entropy production and fluctuations [27].

deal with only few particles or degrees of freedom. Of course, one can consider a system of many *independent* particles, whose behavior is completely determined by the behavior of the one particle system, but this is clearly too unreasonable an idealization to be a realistic model.

While treating really interacting systems is outside the possibilities of present-day technology, it seems now possible to tackle the case of weakly interacting systems. A success in such an endeavor would constitute a fundamental step in the direction of a rigorous foundation of nonequilibrium statistical mechanics. Yet, to this end several ingredients are necessary:

- a refined understanding of the statistical properties of individual systems (*decay of correlations*);
- a detailed understanding of the behavior of such statistical properties under small perturbations (*linear response*);
- a technique to investigate the motion of slowly varying quantities under the influence of fast varying degree of freedom (*averaging*).[3]

Dolgopyat has made fundamental contributions to all the above problems and is currently carrying out a monumental research program to harvest the results of such preliminary successes.[4]

In the rest of this paper, I will discuss results and techniques that constitute these contributions.

2. STATISTICAL PROPERTIES OF FLOWS

Geodesic flows on manifolds of negative curvature represent one of the most interesting classes of dynamical systems. Their hyperbolicity was established by Jacques Hadamard and Élie Cartan (see [36] for details). Ergodicity has been first shown by Eberhard Hopf [29] for special cases and then, in the general setting, by Dmitry Anosov [1]. The mixing is due to Yakov Sinai [2, 49].

Important related systems are the various types of billiards for which hyperbolicity, ergodicity and (possibly) mixing are understood, starting with the work of Sinai [48] further developed an made precise in [9, 13].

What was missing at the closing of last century was a *quantitative understanding* of the rate of mixing (*i.e., decay of correlations*) for geodesic flows and similar systems with hyperbolic behavior.[5] Let us be more precise and a bit more general.[6]

[3]This situation can either consist of variables moving slowly with respect to others, or variables oscillating rapidly and of which one wishes to study only the average behavior.

[4]See, for example, the truly impressive [11, 12] whose results are described in Nikolai Chernov's article in this same issue.

[5]With the notable exception of the geodesic flow on surfaces and some three-manifolds of constant negative curvature where some results could be obtained using techniques of harmonic analysis starting with [14].

[6]For clarity, I will not state the results in full generality, *e.g.,* I will not discuss Axiom-A flows.

Let (M,ϕ_t) be a smooth Anosov flow, that is, M is a Riemannian manifold, ϕ_t is a one parameter group of $\mathscr{C}^r(M,M)$, $r \ge 5$,[7] diffeomorphisms such that

- the tangent bundle $\mathscr{T}M$ has three invariant continuous sections E^s, E^c, and E^u,
- E^c is one-dimensional and tangent to the flow direction
- $\mathscr{T}_xM = E^s(x) \oplus E^c(x) \oplus E^u(x)$ for each $x \in M$
- there exist $C,\lambda > 0$ such that for all $t \in \mathbb{R}_-$
  - if $v \in E^s$, then $\|d\phi_t v\| \ge Ce^{-\lambda t}\|v\|$
  - if $v \in E^c$, then $\|d\phi_t v\| \ge C\|v\|$
- if $t \in \mathbb{R}_+$ and $v \in E^u$, then $\|d\phi_t v\| \ge Ce^{\lambda t}\|v\|$
- the flow is topologically transitive.

Then, denoting by m the Riemannian volume, it is known that there exists a unique measure μ such that for every $g \in \mathscr{C}^0(M,\mathbb{R})$ and m-almost every $x \in M$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g\circ\phi_t(x)\,dt = \int_M g(z)\,\mu(dz) = \mu(g). \tag{2.1}$$

Note that if we consider $\{g\circ\phi_t\}$ as random variables (the randomness being in the initial conditions, distributed according to m), then (2.1) is just the Law of Large Numbers. A measure with the above property is called a *physical measure* and, in the present case, coincides with the celebrated SRB (Sinai–Ruelle–Bowen) measure [8].[8] Of course, in the case of geodesic flows $\mu = m$ and (2.1) corresponds to the aforementioned ergodicity.

The mixing property states that for all $f,g \in \mathscr{C}^0(M,\mathbb{R})$, the following holds:

$$\lim_{t\to\infty} m(f\cdot g\circ\phi_t) = \mu(f)m(g).$$

The long-standing question mentioned above concerns the speed of convergence to such a limit.[9] For more than thirty years no progress whatsoever occurred on this issue until Nikolai Chernov restarted the field by obtaining some partial results [10]. Shortly after, Dolgopyat entered the field with a series of papers that have changed it forever. Let me explain, from my peculiar point of view, the obstacle removed by Dolgopyat's work.

Following a well-established path, let us consider the evolution of the probability measures

$$\mathscr{L}_t\mu(f) = \mu(f\circ\phi_t).$$

[7]The smoothness requirement is not optimal. It is used in Dolgopyat's work to simplify certain arguments.

[8]The SRB measure is a measure whose conditional on unstable manifolds is absolutely continuous with respect to Lebesgue. It is a special case of the so-called u-measures discussed in Section 3.1 of Yakov Pesin's article in this same issue.

[9]Actually, to speak meaningfully of speed of convergence it is necessary to restrict the class of functions under consideration. For example $f,g \in \mathscr{C}^r(M,\mathbb{R})$, $r > 0$, will do.

For each fixed $t \in \mathbb{R}_+$, $\mathscr{L}_t$ is called the *transfer operator* associated to the map ϕ_t.[10] For many reasonable topologies, $\mathscr{L}_t$ is a strongly continuous semigroup.[11] Hence, it has a generator Z, and its resolvent satisfies

$$R(z) := (z\mathbb{1} - Z)^{-1} = \int_0^\infty e^{-zt}\mathscr{L}_t \, dt \tag{2.2}$$

$$\mathscr{L}_t = \lim_{L\to\infty}\int_{-L}^{L} e^{at+ibt} R(a+ib)\, db, \tag{2.3}$$

at least for $\Re(z) = a$ large enough [15]. Clearly the SRB measure of the flow μ satisfies $\mathscr{L}_t\mu = \mu$. The main difference between the transfer operator of a flow and the transfer operator of a map is obvious: in the case of an Anosov map, the dynamics is acting nontrivially in all directions (either expanding or contracting, one being the dual behavior of the other) but in the case of flows there is a direction (the flow direction) in which the dynamics acts trivially. It is then totally unclear which mechanism could produce mixing in the flow direction. The advantage of studying the operators $R(z)$ is that they contain an integration along the flow direction, hence the action in the "bad" direction is *compactified*, *i.e.,* each function gets smoothed out in the flow direction by the application of $R(z)$. Thus such operators are morally similar to the transfer operator of an Anosov map and indeed it is easy to check that they can be studied by the same techniques and enjoy similar properties.

In particular, for all $z \in \mathbb{C}$ and $\Re(z) \geq 0$, there exists $\sigma_z > 0$ such that if $\varphi, \psi \in \mathscr{C}^r(M,\mathbb{R})$, $m(\psi) = 1$, $d\nu_\psi := \psi\, dm$ (where m is Lebesgue measure), then

$$\begin{aligned} &R(z)\nu_\psi(\varphi) = z^{-1}\mu(\varphi) + \hat{R}(z)\nu_\psi(\varphi),\\ &\left|\hat{R}(z)^n\nu_\psi(\varphi)\right| \leq C_{z,\varphi,\psi}(\Re(z)+\sigma_z)^{-n}. \end{aligned} \tag{2.4}$$

Thus, $\hat{R}(z)\nu_\psi(\varphi)$ is analytic in a neighborhood of zero, as a function of z. More precisely, for each $M > 0$ there is $\omega_M > 0$ such that $\hat{R}(z)\nu_\psi(\varphi)$ is analytic in $\left\{z \in \mathbb{C} : \Re(z) \geq 0\right\} \cup \left\{z \in \mathbb{C} : \Re(z) \geq -\omega_M \text{ and } |\Im(z)| \leq M\right\}$.[12]

For $L > M \geq 0$, define a path $\gamma_{L,M} \in \mathscr{C}^0([-L-a-\omega_M, L+a+\omega_M],\mathbb{C})$ by

$$\gamma_{L,M}(s) := \begin{cases} a + i(s - a - \omega_M), & \text{when } L + a + \omega_M \geq s \geq M + a + \omega_M,\\ -M - \omega_M + s + iM & \text{when } M + a + \omega_M \geq s \geq M,\\ -\omega_M + is & \text{when } |s| \leq M,\\ -M - \omega_M - s - iM & \text{when } -M \geq s \geq -M - a - \omega_M,\\ a + i(s + a + \omega_M) & \text{when } -M - a - \omega_M \geq s \geq -L - a - \omega_M. \end{cases}$$

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[10] Historically, the name transfer operator is used for the operator that evolves the densities, *i.e.,* let $\mu \ll m$, $\frac{d\mu}{dm} = h \in \mathscr{C}^r(M,\mathbb{R})$, $r \geq 0$, then one can define the operators $\widetilde{\mathscr{L}}_t : \mathscr{C}^r \to \mathscr{C}^r$ by $d(\mathscr{L}_t\mu) = (\widetilde{\mathscr{L}}_t h)dm$. Given the recent developments of the field it seems more adequate to use the name transfer operator for any operator that evolves density, measures, or more generally, currents since they all bear similar properties.

[11] For example, if one considers it as acting on measures with C^r densities with the $\mathscr{C}^r$ topology (for densities).

[12] Note that the domain of analyticity does not depend on the choice of ψ and ϕ.

Note that the path lies in analyticity domain of $\hat{R}(z)$. Thus, by (2.3),

$$\begin{aligned}\mathscr{L}_t \nu_\psi(\varphi) &= \mu(\varphi) + \lim_{L\to\infty}\int_{-L}^{L} e^{(a+ib)t}\hat{R}(a+ib)\nu_\psi(\varphi)\,db\\ &= \mu(\varphi) + \lim_{L\to\infty}\int_{\gamma_{L,M}} e^{zt}\hat{R}(z)\nu_\psi(\varphi)\,dz\\ &= \mu(\varphi) + \lim_{L\to\infty}\int_{\gamma_{L,M}} \frac{e^{zt}}{z^r}\hat{R}(z)Z^r\nu_\psi(\varphi)\,dz,\end{aligned}$$

where we have used the formula $R(z) = \sum_{k=0}^{r} z^{-k-1}Z^k + R(z)Z^r$, Z being the generator of the flow, and the above analyticity properties. Accordingly,

$$\begin{aligned}\mathscr{L}_t \nu_\psi(\varphi) = \mu(\varphi) + e^{-\omega_M t}&\lim_{L\to\infty}\int_{-M}^{M}\frac{e^{ibt}}{-\omega_M+ib}\hat{R}(-\omega_M+ib)Z^r\nu_\psi(\varphi)\,db\\ &+\lim_{L\to\infty}\int_{\{M\le|b|\le L\}}\frac{e^{at+ibt}}{(a+ib)^r}\hat{R}(a+ib)Z^r\nu_\psi(\varphi)\,db\\ &-\int_{-\omega_M}^{a}\frac{e^{iMt}}{(x+iM)^r}\hat{R}(x+iM)Z^r\nu_\psi(\varphi)\,dx\\ &+\int_{-\omega_M}^{a}\frac{e^{-iMt}}{(x-iM)^r}\hat{R}(x-iM)Z^r\nu_\psi(\varphi)\,dx.\end{aligned}$$

This implies[13]

$$\left|\mathscr{L}_t\nu_\psi(\varphi)-\mu(\varphi)\right| \le C_{M,\varphi,\psi,r}(M^{-r+1}+e^{-\omega_M t}). \tag{2.5}$$

Up to now, we have just rephrased in more modern language results known since the eighties [45]. Yet, in doing so we made clear the nature of the stumbling block: *To make any progress one needs to have a quantitative estimate of the dependence of $C_{M,\varphi,\psi,r}$ and ω_M on M.*

The strongest possible result would be that there exist $\alpha,\omega_* > 0$ such that $\inf_{M\in\mathbb{R}}\omega_M \ge \omega_*$ and $C_{M,\varphi,\psi,r} \le C_{\varphi,\psi,r}M^\alpha$. This, together with (2.5), immediately implies exponential decay of correlations for the flow. We are finally exactly at the core of the Dolgopyat work.

Dolgopyat's inequality. There exist a, α, $\beta > 0$ such that, for each $|b|$ large,

$$R(a+ib)^{\beta\ln|b|}\nu_\psi(\varphi) = \mathcal{O}(|b|^{-\alpha}|\varphi|_s|\psi|_u), \tag{2.6}$$

where $|\varphi|_s = |\varphi|_\infty + |\partial_s\varphi|_\infty$, $\partial_s\varphi$ being the derivative in the strong stable direction, and the analogous definition holds for $|\psi|_u$ with the strong unstable replacing the strong stable. Although we are skipping over many technical details, it is not surprising that such an inequality can be iterated (the point being that the norms on the right-hand side behave well under iteration). Accordingly, by the usual resolvent equalities and the Neumann series for the resolvent we have, for $0<\omega<\frac{\alpha a}{\beta}$,

$$|R(-\omega+ib)\nu_\psi(\varphi)| \le C_{\varphi,\psi}|b|^{\beta\ln(a+\omega)},$$

[13]The constant $C_{M,\varphi,\psi}$ depends on the constants $C_{z,\phi,\psi}$ in (2.4).

and the exponential decay of correlations follows.

The derivation of Dolgopyat's inequality is based on a quantitative version of the *joint nonintegrability* of the strong stable and unstable foliations. The actual proof is rather technical, but it unveils a *new* (non local) mechanism responsible for mixing. This discovery has been the basis of many new results in recent years (*e.g.,* see [3, 4] or the ongoing work of Tsujii that started with [50], just to mention a few).

Let me give a rough idea of why (2.6) holds. First of all, note that by direct computation,

$$R(z)^n \nu_\psi(\varphi) = \frac{1}{(n-1)!}\int_{\mathbb{R}_+} t^{n-1} e^{-zt} \nu_\psi(\varphi\circ\phi_t)\,dt.$$

It is easy to see that the contribution of the integral from zero to cn, for c small enough, is negligible. Next, by the expanding and contracting properties of the dynamics one can assume without loss of generality that φ is essentially constant along the stable fibers and ν_ψ essentially constant along strong unstable leafs. In addition, one can disintegrate ν_ψ along unstable manifolds, thus it suffices to obtain estimates for

$$\frac{1}{(n-1)!}\int_{cn}^{\infty} t^{n-1} e^{-zt}\int_W \varphi\circ\phi_t,$$

where W is a small local strong unstable manifold. By partitioning the time integral in time intervals of fixed length, one is reduced to considering integrals of the type

$$\int_{W_c} e^{-zt}\varphi\circ\phi_l$$

where now W_c is a local central unstable manifold (of a fixed size) and $l \ge cn$. By changing variable, the above integral can be seen as an integral over $\phi_l W_c$ which is a large manifold in the strong unstable direction. Let us partition such a large manifold into manifolds of fixed size $\phi_l W_c = \bigcup_i W_i$. Hence,

$$\int_{W_c} e^{-zt}\varphi\circ\phi_l = \sum_i \int_{W_i} e^{-zt}\varphi J_i, \tag{2.7}$$

where the J_i are determined by the Jacobian of the change of variables and some partition of unity is used to smoothly split the integral.

By the mixing property, the W_i fill all of M. Thus, given any ball U of size comparable to the size of the manifold, it will intersect many W_i. The basic idea is to group the terms of the sum (2.7) according to some covering of M and show that the sum restricted to each single group is small. That is, given U consider the family $\mathscr{W}_U$ of all the W_i that intersect U and let us consider a center unstable manifold W_U going through the "center" of U. We can then use the strong stable holonomy Ψ_i to write all the integrals over such W_i as integrals over W_U. More precisely, let (u,t) the coordinate along the flow and along the strong unstable direction on W_i and (w,s) the corresponding ones on W_U, then we want to perform the change of variables $(u,t) = \Psi_i(w,s)$. Under the *hypothesis that the holonomies are* $\mathscr{C}^1$, we have $t \sim s + a_i \cdot w$.[14] The fact that $a_i \neq 0$ is exactly the *non integrability* assumption of the foliation: if we start from a point in W_U, we go to W_i along the strong stable (holonomy), then move along the strong unstable direction in W_i, then along the strong stable back to W_U and we try to go back to the original point along the strong unstable in W_U, we fail: we find ourself displaced in the time direction. Note that, in the case of contact flows, hence of geodesic flows, an explicit formula for the a_i can be obtained [31]. We can then write

$$\sum_{W_i \in \mathscr{W}_U} \int_{W_i} e^{-zt} \varphi J_i = \sum_{W_i \in \mathscr{W}_U} \int_{W_U} e^{-z(s+a_i w)} \varphi \tilde{J}_i + \mathscr{O}(|\partial_s \varphi|_\infty)$$

where $\tilde{J}_i$ is a $\mathscr{C}^1$ function taking into account all the Jacobians. By the Schwartz inequality, it follows that

$$\sum_{W_i \in \mathscr{W}_U} \int_{W_i} e^{-zt} \varphi J_i = |\varphi|_\infty \sqrt{\sum_{W_i, W_j \in \mathscr{W}_U} \int_{W_U} e^{-z(a_i - a_j) w} \tilde{J}_j \tilde{J}_i} + \mathscr{O}(|\partial_s \varphi|_\infty). \tag{2.8}$$

It is then clear that the integrals under the square root are all of order $|\tilde{J}_j \tilde{J}_i|_{\mathscr{C}^1} \times |z|^{-1} |a_i - a_j|^{-1}$. At this point, it is just a matter to estimate how close two manifolds can typically be. This will allow to obtain the desired result. I do not elaborate this last part of the argument as it does not contain new ideas. The turning point is equation (2.8), where the failure of joint integrability (embodied in the fact that $a_j - a_i \neq 0$) implies that the integrals are much smaller than previously estimated.

REMARK 2.1. The estimates (2.4) (which were the only ones available before Dolgopyat) were totally inadequate since, not taking advantage of the presence of rapidly oscillating functions inside the integrals, they did not provide the factor $|z|^{-1}$ which shows that the integral is smaller for larger imaginary part of z.

By the above argument, and thanks to several highly nontrivial refinements, Dolgopyat has been able to prove:

- Exponential decay of correlations for mixing Anosov flows with $\mathscr{C}^1$ foliations [16];
- Rapid mixing for $\mathscr{C}^\infty$ Axiom A flows with two periodic orbits having periods with a Diophantine ratio [17];
- Generic exponential mixing for suspension over shifts [18];
- Decay of correlation for group extensions (a quantitative version of Brin theory) [19].

3. A NEW APPROACH: STANDARD PAIRS

The results described in the previous section are technically amazing but the proofs are still in the path of the traditional approach to the study of statistical

[14] Where a_i is proportional to the distance between W_U and W_i.

properties of dynamical systems. Indeed, even though I totally underplayed this aspect, Dolgopyat's argument uses Markov Poincaré sections, and the consequent reduction of the system to a symbolic one.[15] Hence, Dolgopyat does not deal directly with the operators described in the previous section but rather with their counterpart for the associated symbolic system. This makes Dolgopyat's original strategy less transparent and immediate, but the substance of the argument is exactly as described above.

The limitations of the Markov partition approach have been felt for long time in the Russian School and have given rise to several alternative approaches to study the ergodic properties of dynamical systems (see *e.g.,* [30] and the references therein). Yet, till the mid nineties no general alternative was available to obtain *quantitative* statistical results (such as estimates on the *decay of correlations*).[16]

In the 1990's many people working on different aspects of dynamical systems deeply felt the need to overcome the traditional approach to studying quantitative statistical properties and to develop a strategy independent of Markov partitions. Due to such a feeling a collective effort took place from the mid nineties onward to devise alternative approaches to the study of the statistical properties of dynamical systems. As a byproduct, today there exist several alternative approaches that can be applied to a variety of systems. One of the most powerful and arguably the most flexible is due to Dolgopyat: *standard pairs*. Indeed, I am convinced that we have not seen yet the full extent of applicability of this approach.[17]

The idea of standard pairs first appeared in [20] where Dolgopyat puts forward a unified approach for the study of limit laws in dynamical systems with some hyperbolicity. That work was also the starting point of Dolgopyat's later study of systems with slow-fast degrees of freedom that I will mention later on. The standard-pairs strategy was then fully developed in [21] in which it was used to prove the *linear response* formula for partially hyperbolic systems. The new element being a new version of *coupling*[18] that is particularly flexible and adapted to the study of systems with some hyperbolicity.

Let me describe briefly the idea in a simple setting. Given a dynamical system (M, f) with a strong unstable foliation (of dimension d_u), one can consider a class $\mathscr{W}$ of smooth manifolds "close" to the unstable foliation. For example,

[15] The presentation in the previous section is phrased in a language akin to [40] although the argument presented are strictly Dolgopyat's.

[16] With the notable exception of the case of one-dimensional expanding maps, see [25, 33, 37].

[17] Some other relevant approaches are (I quote papers that have started the approach, the ensuing work is considerable and there is not point in mentioning it here): hyperbolic metrics [39], Young towers [52], random perturbations [41], renewal theory [47], anisotropic Banach spaces [6].

[18] Coupling has been used for some time in abstract ergodic theory under the name of *joining*, it was introduced by Hillel Furstenberg [26] (see [28] for an account of recent developments). Its use in the study of quantitative decay of correlations has been pioneered by Lai-Sang Young [53] inspired by its use in the field of interacting particle systems [38].

one can consider d_u-dimensional manifolds with uniformly bounded curvature and inner and outer size. In addition, one requires the tangent spaces to be uniformly close to the unstable direction. The key property of such a set being that for each $W \in \mathscr{W}$ and $n \in \mathbb{N}$ there exists a set $\{W_i\} \subset \mathscr{W}$ that is a covering of $f^n W$ with a uniformly bounded number of overlaps. The *standard pairs* are then the elements of the set

$$\Omega_{\alpha,D} = \left\{(W,\varphi):\ W \in \mathscr{W},\ \int_W \varphi = 1,\ \|\ln\varphi\|_{\mathscr{C}^\alpha(W,\mathbb{R})} \le D\right\}.$$

For each $\ell = (W,\varphi)$ and $A \in \mathscr{C}^0(M,\mathbb{R})$, we can write

$$\mathbb{E}_\ell(A) = \int_W A\varphi.$$

Then $\Omega_{\alpha,D}$ can be naturally viewed as a subset of the probability measures on M. Also, we require for standard pairs an extension of the covering property imposed on $\mathscr{W}$. Namely, for each $\ell \in \Omega_{\alpha,D}$ and $n \in \mathbb{N}$ there exist $\{\ell_i\} \subset \Omega_{\alpha,D}$ and $\alpha_i \ge 0$, $\sum_i \alpha_i = 1$ such that, for each $A \in \mathscr{C}^0(M,\mathbb{R})$,

$$\mathbb{E}_\ell(A \circ f^n) = \sum_i \alpha_i \mathbb{E}_{\ell_i}(A). \tag{3.1}$$

Essentially, the above means that the dynamics preserves the regularity of the densities of the measures. This is a natural requirement since the dynamics, restricted to directions close to the strong unstable manifold, is expanding and hence tends to regularize the densities as it happens in expanding maps.

Let $\overline{\Omega}_{\alpha,D}$ be the weak closure of the convex hull of $\Omega_{\alpha,D}$, then (3.1) implies $f_*\overline{\Omega}_{\alpha,D} \subset \overline{\Omega}_{\alpha,D}$. Thus, any invariant measure obtained by a Krylov–Bogoliubov method starting with a measure in $\overline{\Omega}_{\alpha,D}$ must belong to $\overline{\Omega}_{\alpha,D}$. This can be used to prove the existence of the SRB measure for the system (and, more generally, u-measures). Indeed, the above is similar to the approach used in [43]. Yet, here one does not use directly the unstable manifolds (in the same spirit of [39]). As a consequence, the construction is much more flexible. Although this may seem a small change in point of view, the consequences are far-reaching. In particular, this approach is well suited to study the statistical properties of the above invariant measures and of their perturbations.

To this effect a further hypothesis is needed: assume that for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that, for each $W \in \mathscr{W}$, $f^{n_\varepsilon} W$ is ε-close to every point. For example, this is the case if the system is *topologically mixing*.

Accordingly, given any two standard pairs ℓ, ℓ', for n_0 large, we will have many (*i.e.*, a fixed percentage) of the manifolds W_i, W'_j, which constitute the decomposition of $f^n W, f^n W'$, close together. The basic idea is then to match (couple) the mass in nearby leaves along the weak stable foliation. To see how to proceed, let us consider two manifolds W_i and W'_j that are *ε-close*. By this we mean that the weak-stable holonomy Ψ between the two manifolds is well defined on a fixed percentage of their volume and $\operatorname{dist}(\Psi(x), x) \le \varepsilon$.[19] Next, since $(W_i, \varphi_i), (W'_j, \varphi'_j) \in \Omega_{\alpha,D}$, it follows that φ_i, φ'_j are larger, in absolute value, than $e^{-D\delta}$, δ being the uniform size of the elements of $\Omega_{\alpha,D}$. Fix $a \in (0, e^{-D\delta})$ small enough and let $\phi \in \mathscr{C}^\infty(W, \mathbb{R})$ be supported on the domain of Ψ, then by a change of variables,

$$\int_{W_i} A\phi = \int_{W'_j} A \circ \Psi^{-1} \phi \circ \Psi^{-1} J\Psi,$$

where $J\Psi$ is the Jacobian associated to the change of variables. Starting with Anosov's work [1] it is well known that in many cases the holonomy and its Jacobian are Hölder [32], thus $(W_i,\, z_a(\varphi_i - a\phi))$, $\big(W'_j,\, z_a(\varphi_j - a\phi \circ \Psi^{-1} J\Psi)\big) \in \mathscr{D}_{\alpha,D}$ provided that a is chosen small enough, z_a being the normalization factor. On the other hand, $(W_i, a\phi)$ and $(W'_j, a\phi \circ \Psi^{-1} J\Psi)$ represent measures that may not belong to $\mathscr{D}_{\alpha,D}$ but are bound to have the same evolution under the dynamics. Indeed, for each $A \in \mathscr{C}^1(M, \mathbb{R})$ and $n \in \mathbb{N}$,

$$\left| \int_{W_i} A \circ f^n \phi - \int_{W'_j} A \circ f^n \cdot \phi \circ \Psi^{-1} J\Psi \right| = \left| \int_{W_i} \left[A \circ f^n - A \circ f^n \circ \Psi \right] \phi \right| \le C\varepsilon \|A\|_{\mathscr{C}^1} \sigma^n,$$

where $\sigma \le 1$ and is strictly smaller than 1 if the holonomy goes along the strong stable (*e.g.*, in the case of Anosov diffeomorphisms).

Since a fixed proportion, say ρ, of the mass can be matched at any n_0 interval of time,

$$\left| \mathbb{E}_\ell\big(A \circ f^{2kn_0}\big) - \mathbb{E}_{\ell'}\big(A \circ f^{2kn_0}\big) \right| \le \varepsilon \sigma^{-kn_0} |A|_{\mathscr{C}^1} + (1-\rho)^k |A|_{\mathscr{C}^0}.$$

In the easiest possible case ($\sigma < 1$) this immediately implies that all the measures associated to standard pairs converge[20] exponentially fast to the same limiting object, call it μ, which is clearly an invariant measure.

The above approach, presented here in a nutshell, has been remarkably successful in the study of partially hyperbolic systems (see Yakov Pesin's companion paper) and systems with discontinuities (as nicely illustrated in Chernov's contribution to this issue).

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[19] Remember that given two manifolds W, W' the weak-stable holonomy $\Psi\colon W \to W'$ is defined as follows: given $x \in W$ and calling $W^{cs}(x)$ its local weak-stable manifold, $\{\Psi(x)\} = W' \cap W^{cs}(x)$.

[20] By a simple approximation argument one can prove that the convergence takes place also for $A \in \mathscr{C}^0(M, \mathbb{R})$, yet if one wants to have quantitative results it is necessary to use smoother test functions. In other words, we observe exponential decay only if we consider the convergence with respect to the topology of the distributions of order 1 (or $\alpha > 0$), that is we have to regard the measures as elements of $\mathscr{C}^1(M, \mathbb{R})'$.

4. LIMIT THEOREMS: A UNIFIED POINT OF VIEW

Another important contribution by Dolgopyat is the idea to combine the standard-pair technique and the *martingale problem* of Daniel Stroock and Srinivasa Varadhan [42] to develop a general and powerful approach to study *averaging* in Dynamical Systems [22]. Some results that can be obtained by such an approach are detailed and discussed in Chernov's paper in this same issue.

To give a quick idea of the method let us use it to prove something quite obvious: the weak law of large numbers. Let us consider the measure μ constructed in the previous section. One would like to prove that it is a physical measure, *i.e.,* for all $A \in \mathscr{C}^0(M,\mathbb{R})$ and Lebesgue almost all points $x \in M$ we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} A\big(f^k(x)\big) = \mu(A).$$

This is the strong law of large numbers, here we aim at proving the weaker statement: for all $B \in \mathscr{C}^0(\mathbb{R},\mathbb{R})$

$$\lim_{n\to\infty}\int_M B\left(\frac{1}{n}\sum_{k=0}^{n-1} A\circ f^k(x)\right) m(dx) = B(\mu(A)). \tag{4.1}$$

Clearly, it suffices to prove the statement for smooth A, B. Let us define

$$S_{N,t} := \frac{1}{N}\sum_{k=0}^{\lfloor tN\rfloor -1} A\circ f^k + \frac{1}{N} A\circ f^{\lfloor tN\rfloor}\big(tN - \lfloor tN\rfloor\big),$$

where $\lfloor s\rfloor := \max \mathbb{Z}\cap(-\infty, s]$ for $s \in \mathbb{R}$. Then $S_{N,\cdot}(x) \in \mathscr{C}^0(\mathbb{R}_+,\mathbb{R})$ for each $x \in M$. Hence $S_{N,t}$ can be thought as a family of random variables in $\mathscr{C}^0(\mathbb{R}_+,\mathbb{R})$ with law defined by the finite-dimensional distributions determined, for each function $B \in \mathscr{C}^0(\mathbb{R}^k,\mathbb{R})$, by

$$\mathbb{E}\big[B\big(S_{N_1,t_1},\dots,S_{N_k,t_k}\big)\big] = \int_M B\big(S_{N_1,t_1}(x),\dots,S_{N_k,t_k}(x)\big)\, m(dx).$$

Moreover, since

$$S_{N,t+h}(x) - S_{N,t}(x) = \mathscr{O}(h)$$

for each $h \ll 1$, it is immediate that such a family of processes is tight. Hence, there exist accumulation points for $N \to \infty$.

The goal is then to study such accumulation points and prove that they all coincide, thereby proving that the sequence of random variables converges. To this end, let $\overline{B} \in \mathscr{C}^2(\mathbb{R}^2,\mathbb{R})$ and $h \ll 1$. Then,

$$\begin{aligned}
&\overline{B}(S_{N,t+h}(x), t+h) - \overline{B}(S_{N,t}(x), t)\\
&\qquad = \partial_S\overline{B}(S_{N,t}(x), t)\left[\frac{1}{N}\sum_{k=\lfloor (t+h^2)N)\rfloor}^{\lfloor (t+h)N\rfloor -1} A\circ f^k\right] + \partial_t\overline{B}(S_{N,t}(x), t)h + \mathscr{O}(h^2).
\end{aligned}$$

By the mixing properties of the previous section, we have[21]

$$\mathbb{E}\big[\overline{B}(S_{N,t+h}(x),t+h)-\overline{B}(S_{N,t}(x),t)\big]=\mathbb{E}\big[\partial_S\overline{B}(S_{N,t}(x),t)\mu(A)+\partial_t\overline{B}(S_{N,t}(x),t)\big]h \\ +\mathcal{O}(h^2)+\mathcal{O}\big(e^{-\alpha h^2 N}\big),$$

for some $\alpha>0$. Finally, we chose $\overline{B}(z,t)=B(z+\mu(A)(s-t))$ in order to kill the term proportional to h.[22] If $\hat{S}_t$ is any accumulation point of $S_{N,t}$, then

$$\mathbb{E}\big[B\big(\hat{S}_{t+h}+\mu(A)(s-t-h)\big)-B\big(\hat{S}_t+\mu(A)(s-t)\big)\big]=\mathcal{O}(h^2).$$

Summing the above for $t=kh$, where $k=1,\dots,h^{-1}s$ and taking $h\to 0$, yields

$$\mathbb{E}\big(B(\hat{S}_s)\big)=\mathbb{E}(B(\mu(A)s))=B(\mu(A)s),$$

which, for $s=1$, gives (4.1) (the Weak Law of Large Numbers) after a trivial density argument.

The above approach is extremely flexible. For example the reader can easily apply it to obtain the Central Limit Theorem, the only change is that now one needs to expand to third order since the second order in the Taylor expansion gives the main contributions (this is just Ito's formula).

Of course, to apply such a strategy to a given system a lot of extra work may be necessary. This is clearly remarked and illustrated in Chernov's paper in this same issue which describes applications to much more general (even nonstationary) and physically relevant situations. The above strategy also plays a role in the study of the Lyapunov exponents for some of the systems discussed in Pesin's contribution.

5. CONCLUSIONS

Thanks to the above results and ideas Dolgopyat has set the stage for a monumental research program already well underway. Some relevant topics that can be addressed using these techniques are

- study of an heavy particle interacting with light ones,
- limit laws for systems without an invariant probability measure (for example, Lorentz gas),

[21]The point here is that $S_{N,t}$ depends from the trajectory till time t while in the sum we have created a gap of Nh^2. Hence we can substitute the average of the product by the product of the averages if we have a control on multiple correlations. But it is possible to show that the technique described in the previous section allows to obtain results on multiple correlations as well. Indeed, one can consider standard pairs at time Nt obtained by very small manifolds at time zero. Clearly the past history on such standard pairs will be the same for all points so one has a natural way to condition on the past and the same arguments as in the previous chapters imply a decay of correlations due to the gap between tN and $N(t+h^2)$.

[22]This is simply the solution of the equation $\partial_t\overline{B}=-\mu(A)\partial_z\overline{B}$ with *final* condition $\overline{B}(z,s)=B(z)$. Note that such an equation has a $\mathscr{C}^2$ solution, provided that $B\in\mathscr{C}^2$. Of course, if we apply this method to more general situations we will obtain much more complex (non linear) equations and the existence, regularity and uniqueness of the solutions can pose a real challenge. Yet, only quite weak information is needed. For example, for diffusion equations the uniqueness of the solution of the martingale problem suffices [42].

- long time behavior of nonstationary systems (*e.g.,* particles under the action of an external field),
- systems with weak interactions.

As already mentioned several times, Chernov's paper discusses some of the results already achieved along these lines. Here I would like only to conclude going back to my initial remarks.

I claimed that to establish once and for all the relevance of dynamical systems for nonequilibrium statistical mechanics it is necessary to treat systems of many interacting components (*e.g.,* particles). A first step could be to treat systems with very weakly interacting components. Also, I mentioned some outstanding problems that must be overcome to proceed in such a direction. The techniques and the ideas presented in the previous sections address exactly such obstacles and provide powerful tools to remove them. I believe that Dolgopyat work has cleared the road of many of such difficulties and the path is now open to try to treat nontrivial systems relevant for nonequilibrium statistical mechanics. Dolgopyat is already marching along such a path, I am sure that many will follow.

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