PROPOSITION 2.6.10 (Contraction Principle). *Let X be a complete metric space. Under the action of iterates of a contraction $f\colon X \to X$ all points converge with exponential speed to the unique fixed point of f.*

PROOF. As in euclidean space iterating $d(f(x), f(y)) \le \lambda d(x, y)$ gives

$$d(f^n(x), f^n(y)) \to 0 \quad \text{as } n \to \infty,$$

so the asymptotic behavior of all points is the same. On the other hand (2.2.6) shows that for any $x \in X$ the sequence $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence. Thus for any $x \in X$ the limit of $f^n(x)$ as $n \to \infty$ exists if the space is complete, and by (2.2.5) this limit is the same for all x. (2.2.7) shows that it is a fixed point x_0 of f. (Note that uniqueness of the fixed point does not depend on completeness.) □

As in the euclidean case we see that $d(f^n(x), x_0) \le \dfrac{\lambda^n}{1-\lambda} d(f(x), x)$, that is, all orbits converge to x_0 exponentially fast. If x_0 is already known or an estimate in terms of initial data is not required then one can use (2.2.5) to see that $d(f^n(x), x_0) \le \lambda^n d(x, x_0)$ to get the same conclusion in a more straightforward way.

It is at times useful that the Contraction Principle can be applied under weaker hypotheses than the one we used. Indeed, looking at the proof one can see that it would suffice to assume the following property:

DEFINITION 2.6.11. A map f of a metric space is said to be *eventually contracting* if there are constants $C > 0$, $\lambda \in (0, 1)$ such that

$$d(f^n(x), f^n(y)) \le C\lambda^n d(x, y) \tag{2.6.1}$$

for all $n \in \mathbb{N}$.

It is, however, not only possible to reproduce the proof of the Contraction Principle under this weakened hypothesis, but we can find a metric for which such a map becomes a contraction. Indeed, this metric is uniformly equivalent to the original one.

The change of metric that turns an eventually contracting map into a contraction has an analog for maps that are not necessarily contracting, so we prove a useful slightly more general statement.

PROPOSITION 2.6.12. *If $f\colon X \to X$ is a map of a metric space and there are $C, \lambda > 0$ such that $d(f^n(x), f^n(y)) \le C\lambda^n d(x, y)$ for all $x, y \in X$, $n \in \mathbb{N}_0$ then for every $\mu > \lambda$ there exists a metric d_μ uniformly equivalent to d such that $d_\mu(f(x), f(y)) \le \mu d(x, y)$ for all $x, y \in X$.*

PROOF. Take $n \in \mathbb{N}$ such that $C(\lambda/\mu)^n < 1$ and set

$$d_\mu(x, y) := \sum_{i=0}^{n-1} d(f^i(x), f^i(y))/\mu^i.$$

This is called an *adapted* or *Lyapunov metric* for f. The two metrics are uniformly equivalent:

$$d(x, y) \le d_\mu(x, y) \le \sum_{i=0}^{n-1} C(\lambda/\mu)^i d(x, y) \le \frac{C}{1 - (\lambda/\mu)} d(x, y),$$

Note now that

$$d_\mu(f(x),f(y))=\sum_{i=1}^{n}\frac{d(f^i(x),f^i(y))}{\mu^{i-1}}=\mu(d_\mu(x,y)+\frac{d(f^n(x),f^n(y))}{\mu^n}-d(x,y))$$
$$\leq \mu d_\mu(x,y)-(1-C(\lambda/\mu)^n)d(x,y)\leq \mu d_\mu(x,y).$$

□

As an immediate consequence we see that eventually contracting maps can be made contracting by a change of metric because for $\lambda < 1$ as in Definition 2.6.11 we can take $\mu \in (\lambda, 1)$ in Proposition 2.6.12:

COROLLARY 2.6.13. *Let X be a complete metric space and $f\colon X \to X$ an eventually contracting map (Definition 2.6.11). Then under the iterates of f all points converge to the unique fixed point of f with exponential speed.*

Let us point out one of the major strengths of the notion of an eventually contracting map. As we just found, whether or not a map is a contraction can depend on the metric. This is not the case for eventually contracting maps: If a map f satisfies (2.6.1) and d' is a metric uniformly equivalent to d, specifically $md'(x,y)\leq d(x,y)\leq Md'(x,y)$, then

$$d'(f^n(x),f^n(y))\leq Md(f^n(x),f^n(y))\leq MC\lambda^n d(x,y)\leq \frac{MC}{m}\lambda^n d'(x,y).$$

In other words, only the constant C depends on the metric, not the existence of such a constant.

Even without considering smooth maps, as we did in Proposition 2.2.20, the fixed point of a contraction depends continuously on the contraction. This is useful in applications and therefore it is worthwhile to develop this idea further. The natural way to express continuous dependence is to consider families of contractions parametrized by a member of another metric space.

PROPOSITION 2.6.14. *If X, Y are metric spaces, X is complete, $f\colon X\times Y \to X$ a continuous map such that $f_y := f(\cdot, y)$ is λ-contraction for all $y \in Y$, then the fixed point $g(y)$ of f_y depends continuously on y.*

PROOF. Apply

$$d(x,g(y))\leq \sum_{i=0}^{\infty} d(f_y^i(x),f_y^{i+1}(x))\leq \frac{1}{1-\lambda}d(x,f_y(x))$$

to $x=g(y')=f(g(y'),y')$ to get

$$d(g(y),g(y'))\leq \frac{1}{1-\lambda}d(f(g(y'),y'),f(g(y'),y)).$$

□

Exercises

EXERCISE 2.6.1. Show that an open r-ball is an open set.

EXERCISE 2.6.2. Show that any union (not necessarily finite or countable) of open sets is open, and that any intersection of closed sets is closed.

EXERCISE 2.6.3. Consider the set $\mathbb{Z}$ of integers as a metric space with the euclidean metric $d(n,m) = |n-m|$. Describe the balls $\{n \in \mathbb{Z} \mid d(n,0) < 1\}$ and $\{n \in \mathbb{Z} \mid d(n,0) \leq 1\}$. Which of these is open and which is closed?

EXERCISE 2.6.4. Describe all open sets of $\mathbb{Z}$ (with the euclidean metric $d(n,m) = |n-m|$).

EXERCISE 2.6.5. Show that the interior of any set is open and that the closure of any set is closed.

EXERCISE 2.6.6. Show that the boundary of a subset of a metric space is a closed set and that the boundary of an open set is nowhere dense. Conclude that the boundary of a boundary is nowhere dense.

EXERCISE 2.6.7. Decide, with proof, which of the following are complete metric spaces (with the usual metric): $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $[0,1]$.

EXERCISE 2.6.8. Prove that a closed subset of a complete metric space is complete.

Problems for further study

EXERCISE 2.6.9. Suppose that X is a compact metric space and $f\colon X \to X$ is such that $d(f(x), f(y)) < d(x,y)$ for any $x \neq y$

Prove that f has a unique fixed point $x_0 \in I$ and $\lim_{n\to\infty} f^n(x) = x_0$ for any $x \in I$.

EXERCISE 2.6.10. Suppose that X is a complete metric space such that the distance function is bounded by one from above and $f\colon X \to X$ is such that $d(f(x), f(y)) \leq d(x,y) - 1/2(d(f(x), f(y)))^2$.

Prove that f has a unique fixed point $x_0 \in I$ and $\lim_{n\to\infty} f^n(x) = x_0$ for any $x \in I$.

7. Fractals

a. The Cantor set. We next consider a space that is often seen as an oddity in an analysis course, the Cantor set. We will see, however, that sets like this arise naturally and frequently in dynamics and constitute one of the most important spaces we encounter.

1. *Geometric definition*. The *ternary Cantor set* or *middle-third Cantor set* is described as follows. Consider the unit interval $C_0 = [0,1]$ and remove from it the open middle third $(1/3, 2/3)$ to retain two intervals of length $1/3$ whose union we denote by C_1. Apply the same prescription to these intervals, that is, remove their middle thirds. The remaining set C_2 consists of four intervals of length $1/9$ from each of which we again remove the middle third. Continuing inductively we obtain nested sets C_n consisting of 2^n intervals of length 3^{-n} (for a total length of $(2/3)^n \to 0$). The intersection C of all these sets is nonempty (because they are closed and bounded and by Proposition A.1.23) and closed and bounded because all C_n are. It is called the middle-third or ternary Cantor set.

2. *Analytic definition*. It is useful to describe this construction analytically as follows.

LEMMA 2.7.1. *C is the collection of numbers in $[0,1]$ that can be written in ternary expansion (that is, written with respect to base 3 as opposed to base 10) without using 1 as a digit.*