MASS-07; GEOMETRY

FALL 2007

A.Katok

HOMEWORK # 11

November 19, 2007

Due on Friday, November 30

CONTROL PROBLEMS: You should do these problems independently without consulting other students.

44. On a surface with Riemannian metric express curvature at a point through the error term in the area of a disc centered at this point as the radius goes to zero.

45. Show that the curvature at a point p of a surface with Riemannian metric is equal to the limit of the ratio of the difference between the sum of the angles of a small geodesic triangle Δ and π and the area of the triangle Δ when all vertices of Δ converge to p.

Generalize this statement to geodesic polygons.

REGULAR PROBLEMS:

46. Calculate the length of a circle in the hyperbolic plane and the area of a disc. Show that curvature is equal to -1. Show that both the length and the area grow exponentially as functions of radius as radius goes to infinity.

47. Consider the one-sheet hyperboloid $\mathcal{H}$ in $\mathbb{R}^3$ given by the equation $x^2 + y^2 - z^2 = 1$. Prove that through every point of $\mathcal{H}$ pass two straight lines which lies in $\mathcal{H}$. Find equations of those lines using coordinate z as parameter and prove that the lines are geodesics in $\mathcal{H}$.

48. Prove that $\mathcal{H}$ has negative curvature at every point.

Hint: Use definition of curvature from Problem 45 and geodesic quadrilaterals formed by segments of geodesics from Problem 47.

49. Assume the following statement: *Curvature of any convex surface in $\mathbb{R}^3$ is non-negative at every point.* Prove that no compact surface in $\mathbb{R}^3$ has negative curvature.

50. Prove that the *pseudosphere*, the surface of revolution of the curve in the xz-plane given parametrically

$$x = \frac{1}{\cosh t}, \; z = t - \frac{\sinh t}{\cosh t}, \;\; t \geq 0$$

around the z axis, has constant negative curvature -1.

Hint: Introduce coordinates on the pseudosphere in which the Riemannian metric induced from $\mathbb{R}^3$ has the same form as in the upper half-plane model of the hyperbolic plane.