MASS-07; GEOMETRY

FALL 2007

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HOMEWORK # 9

November 2, 2007

Due on Friday, November 9

CONTROL PROBLEM: You should do this problem independently without consulting other students.

36. Consider a Riemannian metric in local coordinates

$$ds^2 = a(x,y)dx^2 + 2b(x,y)dxdy + c(x,y)dy^2.$$

Interpret the following conditions in term of the coefficients of the metric:

(1) The coordinate curves $x = const$ and $y = const$ are orthogonal;
(2) the coordinate curves $x = const$ and $y = const$ form the angle $\pi/4$ at each point;
(3) The area determined by the metric coincides with the usual area $dxdy$.

REGULAR PROBLEMS:

NOTICE: Problem N37 has been replaced.

37. Prove the following formula for the hyperbolic distance between two points z_1 and z_2 in the upper half-plane

$$d(z_1, z_2) = \ln \frac{|z_1 - \bar{z}_2| + |z_1 - z_2|}{|z_1 - \bar{z}_2| - |z_1 - z_2|}.$$

38. Given a smooth function F on a surface S with Riemannian metric one defines (Riemannian) *gradient* ∇F of F as follows:

At any non-critical point x there is unique direction of fastest increase of F i.e a tangent vector $v \in T_xS$ such that the derivative D_vF of F along v, i.e. along any parametrized curve tangent to v, is maximal among all derivatives along tangent vectors of unit length.

(i) Prove this.

Then

$$\nabla F(x) = \begin{cases} D_v F \cdot v, \text{ if } x \text{ is non-critical} \\ 0, \text{ if } x \text{ is critical.} \end{cases}$$

(ii) Prove that ∇F is a smooth vector field orthogonal to the level curves of the function F at all non-ctitical points.

Hint: Use local coordinates.