science is that they describe a system in a deterministic way. This means that for any allowed initial condition there is a solution, which then describes the evolution from that initial condition onward. In addition, determinacy requires that the solution be unique—if it were not then the initial data would not determine the evolution uniquely and the model would have no predictive value.

This issue came up for example, in Section 9.4g. Right now we examine only the basic fact of existence and uniqueness of solutions by itself. While it can conveniently be taken for granted in the sequel, it is appealing to derive it here as yet another application of the Contraction Principle. Obtaining existence of solutions in this way has the advantage that smooth dependence of the fixed point of a contraction on the contraction has beautiful and useful implications about the behavior of solutions of a differential equation as the initial condition is varied: Small changes in initial condition change the solution only slightly.

a. The uniform case. The present use of the Contraction Principle is called *Picard iteration*. It is the first time we use the Contraction Principle in a function space. The idea here is that we can write a differential equation with initial condition as an integral equation and then apply the integral to continuous functions as candidates for solutions. This operation turns out to be a contraction and hence to improve our guesses at a solution iteratively.

THEOREM 9.4.1. *Suppose* $f\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ *a continuous function that is Lipschitz continuous in* $y \in \mathbb{R}^n$ *with Lipschitz constant* M. *Given any* $(a,b) \in \mathbb{R}\times\mathbb{R}^n$ *and* $\delta < 1/M$ *there is a unique solution* $\varphi_{a,b}\colon (a-\delta, a+\delta) \to \mathbb{R}^n$ *of the differential equation* $\dot{y} = f(t,y)$ *with* $\varphi_{a,b}(a) = b$.

PROOF. *The space*. We use a contraction defined on the space of differentiable functions (candidate solutions). Specifically, the hypothesis on f means that $\|f(t,y')-f(t,y)\| \le M\|y'-y\|$ for $t \in \mathbb{R}$, $y, y' \in \mathbb{R}^n$. Consider the set of continuous functions $\varphi\colon [a-\delta, a+\delta] \to \mathbb{R}^n$ and let $\|\varphi\| := \max_{|t-a|\le\delta} \|\varphi(t)\|$. This is a complete metric space by Theorem A.1.13.
The map. We apply the Contraction Principle to the Picard operator defined by

$$\mathcal{P}_{a,b}(\varphi)\,(t) := b + \int_a^t f(x, \varphi(x))\,dx.$$

The contraction property. Note that

$$\|\mathcal{P}_{a,b}(\varphi) - \mathcal{P}_{a,b}(\varphi')\| = \max_{|t-a|\le\delta} \|\int_0^t f(x,\varphi(x)) - f(x,\varphi'(x))\,dx\| \le M\delta\|\varphi-\varphi'\|,$$

that is, $\mathcal{P}_{a,b}$ is a contraction and hence has a unique fixed point. It remains to show that

DELETE

FIGURE 9.4.1. Picard iteration

fixed points of $\mathcal{P}_{a,b}$ are solutions of $\dot{y} = f(t,y)$ with $\varphi(a) = b$ (and vice versa). To that end differentiate the fixed point condition $\varphi_{a,b}(t) = b + \int_a^t f(x, \varphi_{a,b}(x))\,dx$ with respect to t to get $\dot{\varphi}_{a,b}(t) = f(t, \varphi_{a,b}(t))$ by the Fundamental Theorem of Calculus. Evidently fixed points $\varphi_{a,b}$ of $\mathcal{P}_{a,b}$ satisfy $\varphi_{a,b}(a) = b$. To see conversely that solutions are fixed points insert a solution into the fixed point condition and observe that the integrand is $\dot{\varphi}$, yielding a fixed point by the Fundamental Theorem. Thus existence and uniqueness of the fixed point gives existence and uniqueness of solutions. □

In fact, the solutions are defined for all time in this case (Proposition 9.4.7) by piecing together the local ones obtained here.

EXAMPLE 9.4.2. One can explicitly carry out this iteration scheme for the differential equation $\dot{y} = y$, $y(0) = 1$ with $y_0(x) = 1$ as the initial guess. Then $y_1(x) = 1 + \int_0^x y(x)\,dx = 1 + \int_0^x dx = 1 + x$ and $y_2(x) = 1 + \int_0^x 1 + x\,dx = 1 + x + x^2/2$. Inductively, $y_k(x) = \sum_{n=0}^k x^n/n!$, so $y(x) = \sum_{n=0}^\infty x^n/n! = e^x$.

Picard invented this scheme well before the Contraction Principle was available, and this method of successive approximation was carried out by verifying that the errors shrink sufficiently fast.

b. The nonuniform case. It may happen that the Lipschitz constant of the right hand side of the differential equation depends on t and that the right hand side is not even defined for all time and not on all of $\mathbb{R}^n$ either. In that case there is still a result like Theorem 9.4.1, but some care must be taken that the solutions do not leave the domain of the right hand side:

THEOREM 9.4.3. *Suppose $I \in \mathbb{R}$ is an open interval, $O \subset \mathbb{R}^n$ open, $f\colon I \times O \to \mathbb{R}^n$ a continuous function that is an M-Lipschitz continuous function of $y \in O$ for any fixed $t \in I$. Given any $(a, b) \in I \times O$ there exists a $\delta > 0$ such that there is a unique solution $\varphi_{a,b}\colon (a - \delta, a + \delta) \to \mathbb{R}^n$ of the differential equation $\dot{y} = f(t, y)$ with $\varphi_{a,b}(a) = b$.*

PROOF. *The space*. The hypothesis on f means that $\|f(t, y') - f(t, y)\| \le M\|y' - y\|$ for $t \in I$, $y, y' \in O$. Take a closed bounded subset K of O and a closed interval $I' \subset I$ containing a. Let $B > \sup_{t \in I', x \in K} \|f(t, x)\|$ and take $\delta \in (0, 1/M)$ such that $[a - \delta, a + \delta] \subset I'$ and the ball $B(b, B\delta)$ is contained in K. Now consider the set $\mathcal{C}$ of continuous functions $\varphi\colon [a - \delta, a + \delta] \to O$ such that $\|\varphi - b\| < B\delta$, where again $\|\varphi\| := \max_{|t-a| \le \delta} \|\varphi(t)\|$. $\mathcal{C}$ is a closed subset of the complete metric space af all continuous functions on $[a - \delta, a + \delta]$ (with this norm) and hence itself complete.
The map. The Picard operator is again defined by

$$\mathcal{P}_{a,b}(\varphi)\,(t) := b + \int_a^t f(x, \varphi(x))\,dx.$$

Then $\|\mathcal{P}_{a,b}(\varphi) - b\| \le \max_{|t-a| \le \delta} \|\int_a^t f(x, \varphi(x))\,dx\| < B\delta$, so $\mathcal{P}_{a,b}(\varphi) \in \mathcal{C}$ for $\varphi \in \mathcal{C}$, that is, $\mathcal{P}_{a,b}$ is well-defined.
The contraction property. Since

$$\|\mathcal{P}_{a,b}(\varphi) - \mathcal{P}_{a,b}(\varphi')\| = \max_{|t-a| \le \delta} \|\int_0^t f(x, \varphi(x)) - f(x, \varphi'(x))\,dx\| \le M\delta\|\varphi - \varphi'\|,$$

$\mathcal{P}_{a,b}$ is a contraction of $\mathcal{C}$ and hence has a unique fixed point. As before fixed points correspond to solutions. □

REMARK 9.4.4. Note that we only obtain local solutions here. Global ones can be obtained by piecing together local ones; by uniqueness any two local solutions must agree on the intersection of their domains. In fact, the only obstacle to extending solutions is that they may run into the boundary of O, beyond which the ordinary differential equation makes no sense. We carry this out explicitly in Section 9.4g.

c. Continuous dependence. Since $\mathcal{P}_{a,b}$ depends continuously on a and b and $\mathcal{P}_{a,b'}(\mathcal{C}) \subset \mathcal{C}$ for b' sufficiently close to b, the solutions depend continuously on the initial value b by Proposition 2.6.14.

PROPOSITION 9.4.5. *Under the hypotheses of Theorem 9.4.3 solutions depend continuously on the initial value, that is, given $\epsilon > 0$ there exists an $\eta > 0$ such that if $\|b' - b\| < \eta$ then $\max_{|t-a|\le\delta} \|\varphi_{a,b'}(t) - \varphi_{a,b}(t)\| < \epsilon$.*

PROOF. We clearly need to pick η such that $B(b', B\delta) \subset K$ (see the beginning of the previous proof) whenever $\|b' - b\| < \eta$, to make sure that $\varphi_{a,b'}$ is defined for $|t - a| < \delta$. Once this is the case, however, the conclusion (for possibly smaller η) is simply a restatement of the continuous dependence of the fixed point of a contraction on a parameter, in this case with respect to the norm $\|\varphi\| := \max_{|t-a|\le\delta} \|\varphi(t)\|$. □

d. Smooth dependence. The map $\mathcal{P}\colon \mathcal{C} \times \mathbb{R} \times O \to \mathcal{C}$ goes into a linear space, where differentiation makes sense (Definition A.2.1). It depends linearly (hence smoothly) on $b \in O$, and the dependence on $\varphi \in \mathcal{C}$ is through f, and hence as smooth as f. To indicate how one sees this consider the first derivative. The Mean Value Theorem gives

$$\begin{aligned}\mathcal{P}_{a,b}(\varphi)\,(t) - \mathcal{P}_{a,b}(\psi)\,(t) &= \int_a^t f(x, \varphi(x))\,dx - \int_a^t f(x, \psi(x))\,dx \\ &= \int_a^t f(x, \varphi(x)) - f(x, \psi(x))\,dx \\ &= \int_a^t (\partial f/\partial y)(x, c_x)(\varphi(x) - \psi(x))\,dx \\ &\approx \int_a^t (\partial f/\partial y)(x, \varphi(x))(\varphi(x) - \psi(x))\,dx.\end{aligned}$$

The first derivative is thus given by $D\mathcal{P}_{a,b}(\varphi)(\eta)(t) = \int_a^t (\partial f/\partial y)(x, \varphi(x))\eta(x)\,dx$.

Corollary 2.2.15 implies

PROPOSITION 9.4.6. *If in Proposition 9.4.5 the function f is C^r then the solutions are C^{r+1} and depend C^r on the initial value b, that is, $b \mapsto \varphi_{a,b}(a + t)$ is a C^r map for all $t \in (-\delta, \delta)$.*

The fact that the solutions themselves are C^{r+1} follows inductively from the differential equation, which shows that $\dot{y}$ is C^k whenever y and f are C^k.

e. Nonexistence and nonuniqueness. To see that the hypotheses are really needed consider Figure 9.4.2. It shows the solutions $x = ct^2$ for $t\dot{x} = 2x$, where uniqueness fails for the initial condition $a = b = 0$, and existence fails for any initial condition $a = 0$, $b \neq 0$. The right portion of the picture shows that solutions do not always extend to all t where the right hand side $f(t, x)$ is defined for all t: The solutions $x = -1/(t + c)$ of $\dot{x} = x^2$ have singularities for finite t. Existence of solutions can be proved using only

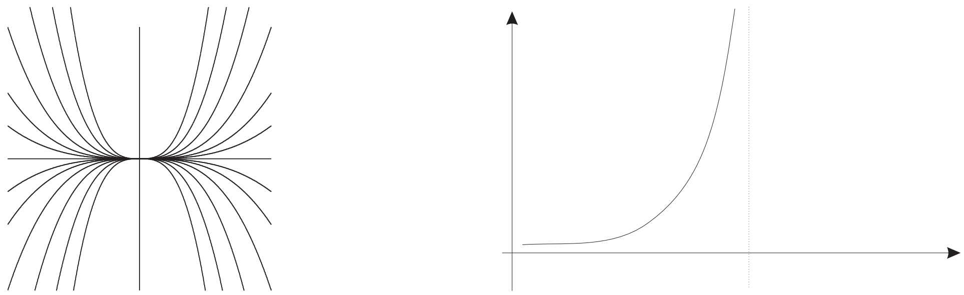

FIGURE 9.4.2. Problems with differential equations

continuity of the right hand side of the differential equation. However, the possible failure of uniqueness shows that continuous dependence on the initial value cannot be expected without a Lipschitz condition.

f. Extension of solutions. For reasons that were previewed in Section 3.2f and are fully justified by Proposition 9.4.11, we restrict attention to differential equations of the form $\dot{x} = f(x)$, that is, differential equations whose right hand side does not depend on time. (These are said to be *autonomous* differential equations and the right hand side is then called the *vector–field* generating the flow.) Physically this reflects the fixed laws of nature we assume. We would prefer not to have to worry about the possibility of solutions being defined only up to some time, and we usually don't:

PROPOSITION 9.4.7. *If f is defined on all of $\mathbb{R}^n$ and is Lipschitz continuous, then the solutions of $\dot{x} = f(x)$ are defined for all t.*

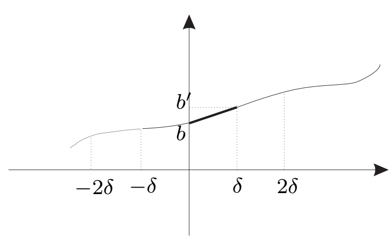

FIGURE 9.4.3. Extension of solutions

PROOF. For any initial condition $y(0) = b$ there is a solution $\varphi_{0,b} \colon [-\delta, \delta] \to \mathbb{R}^n$ with $\varphi_{0,b}(0) = b$ by Theorem 9.4.1. For the initial condition $\varphi_{0,b}(\delta) =: b'$ there is a solution $\varphi_{\delta,\varphi_b(\delta)}$ on $[0, 2\delta]$, that is,

$$\dot{\varphi}_{\delta,\varphi_b(\delta)}(t) = f(\varphi_{\delta,\varphi_b(\delta)}) \text{ and } \varphi_{\delta,\varphi_b(\delta)}(0) = b.$$

At the same time $\dot{\varphi}_{0,b}(t) = f(\varphi_{0,b})$ and $\varphi_{0,b}(0) = b$, so $\varphi_{\delta,\varphi_b(\delta)}(t) = \varphi_{0,b}(t)$ for $t \in [0, \delta]$ by uniqueness. Therefore there is a unique solution on $[-\delta, 2\delta]$. Extending similarly from $-\delta$ gives a solution on $[-2\delta, 2\delta]$, which can in turn be continued to $[-3\delta, 3\delta]$, etc. Thus solutions are defined for all time, independently of the initial condition. $\square$

Applying Proposition 9.4.5 about T/δ times gives:

PROPOSITION 9.4.8. *Solutions depend continuously on the initial value for any finite amount of time, that is, given $T, \epsilon > 0$ there exists a $\delta > 0$ such that if $\|b' - b\| < \delta$ then* $\max_{|t-a|\le T} \|\varphi_{a,b'}(t) - \varphi_{a,b}(t)\| < \epsilon$.

g. Flows. We now study the maps arising from solutions of differential equations.

LEMMA 9.4.9. *The map $\phi_a^t\colon b \mapsto \varphi_{a,b}(a+t)$ of Proposition 9.4.6 is defined on all of $\mathbb{R}^n$ for any value of t, and it is C^r if f is. It is also independent of a.*

PROOF. Proposition 9.4.7 shows that ϕ_a^t is defined on $\mathbb{R}^n$ for any t. Proposition 9.4.6 shows that it is as smooth as f.

Given $a, a' \in \mathbb{R}$ and $b \in \mathbb{R}^n$ consider the solutions $\varphi_{a,b}$ and $\varphi_{a',b}$ of the differential equation. Then $\phi_a^t(b) = \varphi_{a,b}(a+t)$ and $\phi_{a'}^t(b) = \varphi_{a',b}(a'+t)$. We need to show that these coincide. If we define $\psi(t) := \varphi_{a,b}(t+a)$ and $\psi'(t) := \varphi_{a',b}(t+a')$ then we have

$$\dot\psi(t) = f(\psi(t)), \quad \psi(0) = b \quad \text{and} \quad \dot\psi'(t) = f(\psi'(t)), \quad \psi'(0) = b.$$

By uniqueness $\phi_{a'}^t(b) = \varphi_{a',b}(t+a') = \psi'(t) = \psi(t) = \varphi_{a,b}(a+t) = \phi_a^t(b)$. □

We drop the subscript a henceforth and write $\phi^t(b) = \varphi_{a,b}(a+t)$ from now on (and make $a = 0$ our default choice).

DEFINITION 9.4.10. A family $(\phi^t)_{t\in\mathbb{R}}$ of maps for which $(t,x) \mapsto \phi^t(x)$ is C^r is said to be a C^r *flow* if $\phi^{s+t} = \phi^s \circ \phi^t$ for all $s, t \in \mathbb{R}$.

This "group property" holds in our situation:

PROPOSITION 9.4.11. *A differential equation $\dot x = f(x)$ with $f\colon \mathbb{R}^n \to \mathbb{R}^n$ a C^r function and $\|Df\|$ bounded defines a C^r flow on $\mathbb{R}^n$.*

PROOF. Given $t \in \mathbb{R}$ the functions $\psi_1(s) := \varphi_{0,b}(s+t)$ and $\psi_2(s) := \varphi_{0,\varphi_{0,b}(t)}(s)$ are solutions of the differential equation and $\psi_2(0) = \varphi_{0,\varphi_{0,b}(t)}(0) = \varphi_{0,b}(t) = \psi_1(0)$, so $\psi_1 = \psi_2$ by uniqueness. Consequently

$$\phi^s \circ \phi^t(b) = \phi^s(\varphi_{0,b}(t)) = \varphi_{0,\varphi_{0,b}(t)}(s) = \varphi_{0,b}(s+t) = \phi^{s+t}(b).$$

Taking $s = -t$ shows in particular that ϕ^t is invertible with inverse ϕ^{-t}. Thus these maps ϕ^t are C^r diffeomorphisms. □

Section 2.4a gave a complete description of the dynamics of the flow generated by the differential equation $\dot x = f(x)$ on the line, where f is a Lipschitz continuous function: There is a closed set of fixed points and the flow is monotone on every complementary interval with all orbits asymptotic to one endpoint and asymptotic to the other endpoint in negative time.

Changing the size of the right hand side without changing the direction does not change orbits, only the speed along them.

DEFINITION 9.4.12. The flows generated by $\dot x = f(x)$ and $\dot x = a(x)f(x)$ for some continuous nowhere zero scalar function a are said to be related by a *time change*.