MASS-07; GEOMETRY

FALL 2007

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QUESTIONS FOR THE FINAL EXAM

Material to be covered fully or partially in the lectures after the Thanksgiving break is in the brackets [].

1. Various ways of representing surfaces: equations, parametric, factor-spaces, domains with identifications. Give examples covering together all topological types of compact surfaces, explain real and complex smooth structure and a Riemannian metric for some examples.

2. Embedding of surfaces into $\mathbb{R}^3$ and higher-dimensional spaces. Examples of all closed orientable surfaces given by a single equation in $\mathbb{R}^3$. Parametric embedding of Mobius strip. Embedding of the projective plane and Klein bottle into $\mathbb{R}^4$.

3. Classification of orientation preserving isometries of Euclidean plane and (round) sphere. Matrix representation for both groups of isometries using real and complex matrices.

4. Classification of orientation reversing isometries of Euclidean plane and (round) sphere. Classification of isometries of the elliptic plane.

5. Area of a geodesic triangle on the sphere and elliptic plane through angular excess and through the lengths of its sides.

6. Locally and globally symmetric spaces. Transitivity of the isometry group for a globally symmetric space. Euclidean, elliptic and hyperbolic planes as globally symmetric spaces. transitivity of isometry group on unit tangent vectors. Examples of locally symmetric spaces which are not globally symmetric.

7. Definition of a (topological) manifold. Every connected topological manifold is path connected. Example of a compact connected metric space which is not path connected. Real line is not homeomorphic to $\mathbb{R}^n$ for $n \geq 2$.

8. Definition of simplices, triangulation and simplicial complexes. Triangulation of surfaces. Barycentric subdivision of a triangulation Orientability of a triangulation. Maps of surfaces. Definition of Euler characteristic for a triangulation and a map. Proposition: every polygon can be triangulated. Invariance of Euler characteristic under barycentric subdivision and triangulation of faces for a map.

9. Theorem: All triangulations and maps on a given surface have the same Euler characteristics,.

10. Attaching handles and Mobius caps to a surface. Effect on Euler characteristic. Proposition: For a non-orientable surface attaching a handle, a pair of Mobius caps or an inverted handle produce homeomorphic surfaces. Examples of orientable and non-orientable surfaces with all possible values of Euler characteristic. Orientable double cover of a non-orientable surface.

11. Theorem: Given a compact, closed (without boundary), orientable surface M which has a map, there exists an integer $m \geq 0$ such that M is homeomorphic to the sphere with m handles.

12. Classification of non-orientabe surfaces. Every compact closed non-orientable surface which has a map is homeomorphic to a sphere with $m \geq 1$ Mobius caps.

13. Definition of chain complex, its homology groups and Betti numbers. Chain complex associated with a triangulation of a surface. Expression of Euler characteristic through Betti numbers.

14. Interpretation of Betti numbers β_0, β_1, and β_2 for a closed compact surface. Torsion in the homology of non-orientable surfaces.

15. Definition of differentiable manifold. Notions of smooth function and diffeomorphism. Diffeomorphism between $\mathbb{R}^2$, open discs and open rectangles.

16. Partition of unity. Existence of a smooth partition of unity for any finite atlas in a compact surface.

17. Different methods of constructing atlases: coordinate projections, tangent space projections, glueing, quotient spaces. Examples of smooth structures on all surfaces form the standard list: the sphere with handles and Mobius caps.

18. Complex manifolds. Riemann surfaces. Examples: Riemann sphere, flat torus, complex structure on the sphere with m handles.

19. Difference between real and complex (holomorphic) diffeomorphisms. Conformal property of holomorphic functions. Unit disc is not holomorphically equivalent to $\mathbb{C}$. [Examples of flat tori which are not holomorphically equivalent.]

20. Differentiable functions on smooth surfaces. Normal form of a function near a non-critical point. Hessian at a critical point. Non-degenerate critical points: maxima, minima and saddles. [Morse Lemma.]

21. Theorem: For any Morse function $f : S \to \mathbb{R}$, the Euler characteristic is related to the number of critical points by the formula

$$\chi = (\#\text{ of maxima}) - (\#\text{ of saddles}) + (\#\text{ of minima})$$

22. Degree of a circle map. Degree is a complete homotopy invariant for circle maps. Index of an isolated zero (critical point) of a vector field. Independence from a choice of curve. Morse index of a non-degenerate critical point of a smooth function is equal to index of its gradient in any coordinate system.

23. [Theorem: Sum of the indices of critical points of any vector filed on a compact closed surface with finitely many fixed points is equal to Euler characteristic.]

24. Tangent vectors (different definitions and their equivalence), tangent bundle. Atlas in a tangent bundle associated to a given atlas on a surface. Riemannian metrics; existence of a smooth Riemannian metric on any compact smooth surface (use existence of partition of unity). Length, angles and area defined by a Riemannian metric.

25. Geodesics. Existence and uniqueness for a given point and direction (no proof) and between two nearby points. Examples of non-uniqueness: round sphere, flat torus. Geodesic polar coordinates. Examples: Euclidean plane, round sphere, hyperbolic plane.

26. Two conformal models of hyperbolic plane. Geodesics. Intersecting, parallel and ultra-parallel lines. Circles, horocycles and equidistant curves. Representations of these curves in the upper half-plane and disc models.

27. Proposition: If S is a surface endowed with a Riemannian metric such that any two points determine a unique geodesic connecting them, then any isometry I of S is uniquely determined by the images of three points which do not lie on the same geodesic.

Cases of Euclidean, elliptic and hyperbolic planes.

28. Orientation preserving isometries of the hyperbolic plane. The distance via cross-ratio. Characterization of fractional linear transformations and isometries of hyperbolic plane via preservation of cross ratio.

29. Classification of orientation preserving isometries of hyperbolic plane: rotations, parabolic and hyperbolic isometries. Classification up to conjugacy in the isometry group. Relation with three types of curves and three types of geodesic pencils.

30. Theorem: Given a hyperbolic triangle Δ with angles α, β, and γ, the area A of Δ is given by

$$A = \frac{1}{-\kappa}(\pi - \alpha - \beta - \gamma)$$

where κ is the curvature, whose value is -1 for the standard upper half-plane and open disc models.

Equality of geodesic triangles with pairwise equal angles.

31. Construction of hyperbolic metrics of all orientable surfaces of genus ≥ 2. [Associated discrete groups of isometries and tesselations.]

32. Theorem: Let S be a surface with a locally hyperbolic metric (that is, a surface with a metric which is locally isometric to patches of H^2), and let $A(S)$ denote the total area of S. Then

$$A(S) = -2\pi\chi(S).$$

33. Definition of curvature for an arbitrary Riemannian metric through the error term in the circle length. Calculation for the Euclidean plane, round sphere and hyperbolic plane. Equivalence with the definition with the asymptotics of the angular defect of a small geodesic triangle (using Gauss–Bonnet theorem)

34. Gauss–Bonnet Theorem: Let A, B, and C be the vertices of a geodesic triangle Δ on a surface S, and let α, β, and γ be the angles at these vertices. Then the integral of the curvature of S over Δ is equal to the angular excess:

$$\int_\Delta \kappa \, dS = \alpha + \beta + \gamma - \pi.$$

35. [Index of a closed curve with respect to a point. The fundamental theorem of algebra: every polynomial with complex coefficients has a complex root.]

36. [Jordan Curve Theorem for smooth curves: the complement to every smooth closed curve on a plane or a sphere without critical points and self-intersections has exactly two connected components. Application: genus of a smooth surface is equal to the maximal number of disjoint smooth closed curves such that the complement to their union is connected.]