MASS-07; GEOMETRY

FALL 2007

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FINAL EXAM TICKETS

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N1

1.State and prove classification of isometries of Euclidean plane.

2. Prove that every ellipsoid has at least three closed geodesics without self-intersections.

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N2

1. State and prove classification of isometries of (round) sphere and elliptic plane.

2. The arc of the circle $|z - i|^2 = 2$ in the upper half-plane represents an r- equidistant curve, i.e. one of the two connected components of the locus of points at distance r from a certain geodesic. Find r.

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N3

1. State and prove the formula for the area of a geodesic triangle on the sphere and elliptic plane through its angles.

2. Does there exist a covering map from the sphere with 10 handles to
   (i) the sphere with three handles,
   (ii) the sphere with four handles?

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N4

1. Define Euler characteristic for a triangulation and a map.
   Prove that every polygon can be triangulated.
   Prove that Euler characteristic in invariant under barycentric subdivision of a triangulations and triangulation of faces for a map.

2. There are exactly two different horocycles which pass through two given points in the hyperbolic plane.

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N5

1.Prove that all triangulations and maps on a given surface have the same Euler characteristic.

2. A *triply asymptotic triangle* in the hyperbolic plane is a configuration of three lines such that each of them is parallel to the other two in different directions.
   Find the symmetry group of a triply asymptotic triangle.

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N6

1. Prove that for a non-orientable surface attaching a handle, a pair of Möbius caps, or an inverted handle produce homeomorphic surfaces.

2. Consider a disc with k holes D and a vector field in it which is non-vanishing and transversal (non-tangential) at all boundary points.
   For what values of k any such vector field has zeroes inside the domain D?

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N7

1. Prove that any compact, closed (without boundary), orientable surface which has a map, is homeomorphic to the sphere with $m \geq 0$ handles.

2. Prove that for any pairs of points x, y in the elliptic plane there are at most two shortest curves between x and y.

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N8

1.Prove that every compact closed non-orientable surface which has a map, is homeomorphic to a sphere with $m \geq 1$ Möbius caps.

2. Suppose a smooth surface S in $\mathbb{R}^3$ is symmetric with respect to a plane L. Then every connected component of the intersection $S \cap L$ is a geodesic in S.

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N9

1. Define chain complex, its homology groups and Betti numbers. Describe chain complex associated with a triangulation of a surface.

State and prove expression of Euler characteristic through Betti numbers.

2. A *triply asymptotic triangle* in the hyperbolic plane is a configuration of three lines such that each of them is parallel to the other two in different directions. Prove that all triply asymptotic triangles are isometric.

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N10

1. Define diffeomorphism of differentiable manifolds.
   Prove that $\mathbb{R}^2$, open discs and open rectangles are diffeomorphic.

2. Calculate the area of a disc of radius r in the hyperbolic plane.

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N11

1.Prove existence of a smooth partition of unity for any finite smooth atlas in a compact smooth surface.

2. In the elliptic plane find all pairs of points x, y such that the shortest curve between x and y is not unique.

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N12

1.Define Riemann surfaces. Prove that any Riemann surface is orientable. Give examples (with proofs) of a complex structure on all compact orientable surfaces.

2. Let $F(x, y, z)$ be a differentiable function even in z, i.e. $F(x, y, z) = F(x, y, -z)$ and such that 0 is not a critical value. Prove that every connected component of the curve $F(x, y, z) = 0$; $z = 0$ is a geodesic on the surface $F = 0$.

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N13

1. Define real and complex (holomorphic) diffeomorphisms. Prove conformal property of holomorphic functions.Prove that unit disc is not holomorphically equivalent to $\mathbb{C}$.

2. Calculate lengths of all closed geodesics on the standard flat torus.

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N14

1. Define Morse function. Prove that for any Morse function on a surface S the Euler characteristic is related to its critical points by the formula

$$\chi(S) = (\# \text{ of maxima}) - (\# \text{ of saddles}) + (\# \text{ of minima})$$

2. An r- equidistant curve in the hyperbolic plane is one of the two connected components of the locus of points at distance r from a certain geodesic.

For any given $r > 0$ there are exactly two different r-equidistant curves which pass through two given points in the hyperbolic plane.

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N15

1. Define degree of a circle map. Prove that circle maps are homotopic if and only if they have the same degree.

Define index of an isolated zero (critical point) of a vector field. Formulate the connection between indices of critical points of a vector field and Euler characteristic.

2. Let $\ell(r)$ be the length of a circle of radius r in the hyperbolic plane. FInd
$$\lim_{r \to \infty} \frac{\log \ell(r)}{r}.$$

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N16

1. Define Riemannian metric. Explain how length, angles and area are defined by a Riemannian metric.

Prove that a smooth Riemannian metric exists on any compact smooth surface.

2. Let M be the unique compact surface with Euler characteristic -1 and S be any surface with negative Euler characteristics. Prove that there exists a covering map $S \to M$.

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N17

1. Define two conformal models of hyperbolic plane. Define circles, horocycles and equidistant curves.
Show with proofs how these curves are represented in the upper half-plane and disc models.

2. Let $F(x, y, z)$ be a differentiable function even in z, i.e. $F(x, y, z) = F(x, y, -z)$ and such that 0 is not a critical value. Prove that every connected component of the curve $F(x, y, z) = 0$; $z = 0$ is a geodesic on the surface $F = 0$.

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N18

1. Prove: If S is a surface endowed with a Riemannian metric such that any two points determine a unique geodesic connecting them, then any isometry I of S is uniquely determined by the images of three points which do not lie on the same geodesic.

2. Does there exist a covering map from the sphere with 9 handles to
   (i) the sphere with four handles,
   (ii) the sphere with five handles?

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N19

1. Express the distance in the models of hyperbolic plane via cross-ratio with proof.

Characterize fractional linear transformations and isometries of hyperbolic plane via preservation of cross ratio with proof.

2. Prove that every ellipsoid has at least three closed geodesics without self-intersections.

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N20

1. State and prove classification of orientation preserving isometries of hyperbolic plane: rotations, parabolic and hyperbolic isometries.

Explain with proof relation with three types of curves and three types of geodesic pencils.

2. Consider two connected smooth curves in $\mathbb{R}^2$. Prove that their direct product in $\mathbb{R}^4$ is isometric to one of the three surfaces: a flat torus, a flat cylinder, or $\mathbb{R}^2$.

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N21

1.State and prove the formula for the area of a geodesic triangle in the hyperbolic plane through its angles.

2. Suppose a smooth surface S in $\mathbb{R}^3$ is symmetric with respect to a plane L. Then every connected component of the intersection $S \cap L$ is a geodesic in S.

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N22

1. Prove that in the hyperbolic plane geodesic triangles with pairwise equal angles are isometric.

2. Let S be a surface in $\mathbb{R}^3$, $p \in S$ and degree of tangency between S and its tangent plane at p is greater than one, i.e. the distance between a point on the surface at a distance r from p and its projection to the tangent plane is $O(r^3)$.
   Prove that curvature of S at p is equal to zero.

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N23

1. Describe a construction of hyperbolic metrics of all orientable surfaces of genus ≥ 2 with proofs. Calculate the area of those surfaces with respect to the hyperbolic metric.

2. Assume a smooth surface S in $\mathbb{R}^3$ is symmetric with respect to a rotation R by π around an axis which intersects S at an isolated point p. Show that R acts as the geodesic flip near p, i.e. it keeps any geodesic passing through p invariant and reflects it at p preserving the length parameter.

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N24

1. Let S be a surface with a locally hyperbolic metric (that is, a surface with a metric which is locally isometric to patches of H^2), and let $A(S)$ denote the total area of S. Prove that

$$A(S) = -2\pi\chi(S).$$

2. In the elliptic plane find all pairs of points x, y such that the shortest curve between x and y is not unique.

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N25

1.Defines curvature for a surface with Riemannian metric. Let Δ be a geodesic triangle on a surface S, with angles α, β, and γ.

Prove that the integral of the curvature of S over Δ is equal to the angular excess:

$$\int_\Delta \kappa \, dS = \alpha + \beta + \gamma - \pi.$$

2. The arc of the circle $|z - 2i|^2 = 8$ in the upper half-plane represents an r- equidistant curve, i.e. one of the two connected components of the locus of points at distance r from a certain geodesic. Find r.

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N26

1. Define index of a closed curve with respect to a point.

Prove the fundamental theorem of algebra: every polynomial with complex coefficients has a complex root.

2. Let $A(r)$ be the area of a disc of radius r in the hyperbolic plane. FInd

$$\lim_{r \to \infty} \frac{\log A(r)}{r}.$$

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N27

1.Prove Jordan Curve Theorem for smooth curves: the complement to every smooth closed curve on a plane or a sphere without critical points and self-intersections has exactly two connected components.

2. Consider a line in the hyperbolic plane and a doubly asymptotic triangle for which this line is one of the sides. Assume the angle at the finite vertex is fixed. Find the locus of all finite vertices.

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N28

1.Define genus of a smooth surface through Euler characteristic.
   Prove that it is equal to the maximal number of disjoint smooth closed curves such that the complement to their union is connected.

2. There are exactly two different horocycles which pass through two given points in the hyperbolic plane.

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