$z_1\bar{w}_1 - z_2\bar{w}_2$, i.e.

$$SU(1,1) = \{g \in SL(2,\mathbb{C}) \mid g = \begin{pmatrix} a & c \\ \bar{c} & \bar{a} \end{pmatrix}\},$$

and

$$K = \{\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}\}.$$

The homogeneous space G/K can be identified with the "projectivized" space of the *negative* vectors in $\mathbb{C}^2$ ($< z, z >< 0$), analogous to that discussed above for $\mathbb{R}^3$, or in homogeneous coordinates, with the unit disc in $\mathbb{C}$,

$$\mathcal{U} = \{z \in \mathbb{C} \mid |z| < 1\}.$$

In the Poincaré upper half–plane model, $G = SL(2,\mathbb{R})$, and $K = SO(2)$, the stabilizer of the point $i \in \mathcal{H}$. Here the homogeneous space G/K is identified with the upper half–plane

$$\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}.$$

by the following construction. Each matrix in $SL(2,\mathbb{R})$ can be written as a product of upper–triangular and orthogonal (the Iwasawa decomposition):

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix},$$

where $x, y \in \mathbb{R},\ y > 0$. Then $\pi : G/K \to \mathcal{H}$ given by

$$\pi(g) = g(i) = \frac{ai + b}{ci + d} = x + iy = z$$

does the identification.

In the last two conformal models, the corresponding G acts by fractional–linear transformations: for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $g(z) = \frac{az+b}{cz+d}$.

1.2. The hyperbolic plane

Let $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im(z)} > 0\}$ be the upper–half plane. We have seen (Exercise 2) that equipped with the metric

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}, \tag{1.2.1}$$

it becomes a model of the *hyperbolic* or *Lobachevski plane*. We will see that the *geodesics* (i.e., the shortest curves with respect to this metric) will be straight lines and semicircles orthogonal to the real line

$$\mathbb{R} = \{z \in \mathbb{C} \mid \mathrm{Im(z)} = 0\}.$$

Using this fact and elementary geometric considerations, one easily shows that any two points in $\mathcal{H}$ can be joined by a unique geodesic, and that from any point in $\mathcal{H}$ in any direction one can draw a geodesic. We will measure the distance between two points in $\mathcal{H}$ along the geodesic connecting them.

It is clear that any geodesic can be continued indefinitely, and that one can draw a circle centered at a given point with any given radius.

The tangent space to $\mathcal{H}$ at a point z is defined as the space of tangent vectors at z. It has the structure of a 2–dimensional real vector space or of a 1–dimensional complex vector space: $T_z\mathcal{H} \approx \mathbb{R}^2 \approx \mathbb{C}$. The Riemannian metric (1.2.1) is induced by the following inner product on $T_z\mathcal{H}$: for $\zeta_1 = \xi_1 + i\eta_1$ and $\zeta_2 = \xi_2 + i\eta_2$ in $T_z\mathcal{H}$, we put

$$\langle \zeta_1, \zeta_2 \rangle = \frac{(\zeta_1, \zeta_2)}{\mathrm{Im(z)}^2}, \tag{1.2.2}$$

which is a scalar multiple of the Euclidean inner product $(\zeta_1, \zeta_2) = \xi_1\xi_2 + \eta_1\eta_2$.

We define the *angle* between two geodesics in $\mathcal{H}$ at their intersection point z as the angle between their tangent vectors in $T_z\mathcal{H}$. Using the formula

$$\cos\varphi = \frac{\langle \zeta_1, \zeta_2 \rangle}{\|\zeta_1\|\|\zeta_2\|} = \frac{(\zeta_1, \zeta_2)}{|\zeta_1||\zeta_2|},$$

where $\| \; \|$ denotes the norm in $T_z\mathcal{H}$ corresponding to the inner product $\langle \, , \, \rangle$, and $| \; |$ denotes the norm corresponding to the inner product $(\, , \,)$, we see that this notion of angle measure coincides with the Euclidean angle measure.

The first four axioms of Euclid hold for this geometry. However, the fifth postulate of Euclid's *Elements*, the axiom of parallels, does not hold: there is more than one geodesic passing through the point z not lying in the geodesic L that does not intersect L (see Fig. 1.2.1).

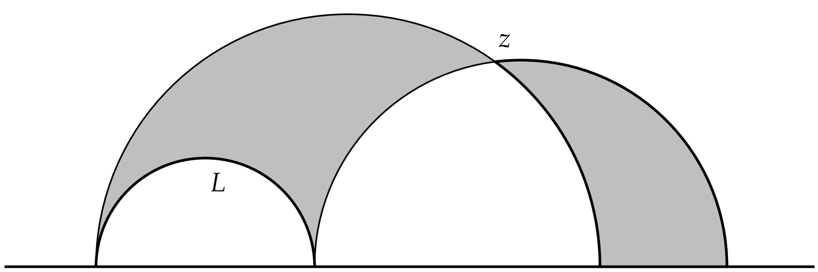

FIGURE 1.2.1. Geodesics in the upper half-plane

Therefore the geometry in $\mathcal{H}$ is *non–Euclidean*. The metric in (1.2.1) is said to be the *hyperbolic metric*. It can be used to calculate the length of curves in $\mathcal{H}$ the same way the Euclidean metric $\sqrt{dx^2 + dy^2}$ is used to calculate the length of curves on the Euclidean plane. Let $I = [0, 1]$ be the unit interval, and $\gamma : I \to \mathcal{H}$ be a piecewise differentiable curve in $\mathcal{H}$,

$$\gamma(t) = \{v(t) = x(t) + iy(t) \mid t \in I\}.$$

The length of the curve γ is defined by

$$h(\gamma) = \int_0^1 \frac{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}}{y(t)} dt. \tag{1.2.3}$$

We define the *hyperbolic distance* between two points $z, w \in \mathcal{H}$ by setting

$$\rho(z, w) = \inf h(\gamma),$$

where the infimum is taken over all piecewise differentiable curves connecting z and w.

PROPOSITION 1.2.1. *The function $\rho : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ defined above is a distance function, i.e., it is*

(a) *nonnegative: $\rho(z, z) = 0$; $\rho(z, w) > 0$ if $z \neq w$;*
(b) *symmetric: $\rho(u, v) = \rho(v, u)$;*
(c) *satisfies the triangle inequality: $\rho(z, w) + \rho(w, u) \geq \rho(z, u)$.*

PROOF. It is easily seen from the definition that (b), (c), and the first part of property (a) hold. The second part follows from Exercise 3. □

Consider the group $SL(2, \mathbb{R})$ of real 2×2 matrices with determinant one. It acts on $\mathcal{H}$ by *Möbius transformations* as follows. To each $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, we assign the transformation

$$T_g(z) = \frac{az + b}{cz + d}. \tag{1.2.4}$$

PROPOSITION 1.2.2. *Any Möbius transformation T_g maps $\mathcal{H}$ into itself.*

PROOF. We can write

$$w = T_g(z) = \frac{(az + b)(c\overline{z} + d)}{|cz + d|^2} = \frac{ac|z|^2 + adz + bc\overline{z} + bd}{|cz + d|^2}.$$

Therefore

$$\mathrm{Im(w)} = \frac{\mathrm{w} - \overline{\mathrm{w}}}{2\mathrm{i}} = \frac{\mathrm{(ad - bc)(z - \overline{z})}}{2\mathrm{i}|\mathrm{cz + d}|^2} = \frac{\mathrm{Im(z)}}{|\mathrm{cz + d}|^2}. \tag{1.2.5}$$

Thus $\mathrm{Im(z)} > 0$ implies $\mathrm{Im(w)} > 0$. □

One can check directly that if $g, h \in SL(2, \mathbb{R})$, then $T_g \circ T_h = T_{gh}$ and $T_g^{-1} = T_{g^{-1}}$. It follows that each T_g, $g \in SL(2, \mathbb{R})$ is a bijection, and thus we obtain a *representation* of the group $SL(2, \mathbb{R})$ by Möbius transformations of the upper–half plane $\mathcal{H}$. In fact, the two matrices g and $-g$ give the same Möbius transformation, so formula (1.2.4) actually gives a representation of the quotient group $SL(2, \mathbb{R})/\{\pm 1_2\}$ (where 1_2 is the 2×2 identity matrix) denoted by $PSL(2, \mathbb{R})$, which we will identify with the group of Möbius transformations of the form (1.2.4). Notice that $PSL(2, \mathbb{R})$ contains all transformations of the form

$$z \to \frac{az + b}{cz + d} \quad \text{with} \quad ad - bc = \Delta > 0,$$

since by dividing the numerator and the denominator by $\sqrt{\Delta}$, we obtain a matrix for it with determinant equal to 1. In particular, $PSL(2,\mathbb{R})$ contains all transformations of the form $z \to az + b$ ($a, b \in \mathbb{R}$, $a > 0$). Since transformations in $PSL(2,\mathbb{R})$ are continuous, we have the following result.

THEOREM 1.2.3. *The group $PSL(2,\mathbb{R})$ acts on $\mathcal{H}$ by homeomorphisms.*

DEFINITION. A transformation of $\mathcal{H}$ onto itself is called an *isometry* if it preserves the hyperbolic distance in $\mathcal{H}$.

Isometries clearly form a group; we will denote it by $\mathrm{Isom}(\mathcal{H})$.

THEOREM 1.2.4. *Möbius transformations are isometries, i.e., we have the inclusion $PSL(2,\mathbb{R}) \subset \mathrm{Isom}(\mathcal{H})$.*

PROOF. Let $T \in PSL(2,\mathbb{R})$. By Theorem 1.2.3 T maps $\mathcal{H}$ onto itself. Let $\gamma : I \to \mathcal{H}$ be the piecewise differentiable curve given by $z(t) = x(t) + iy(t)$. Let

$$w = T(z) = \frac{az+b}{cz+d};$$

then we have $w(t) = T(z(t)) = u(t) + iv(t)$ along the curve γ. Differentiating, we obtain

$$\frac{dw}{dz} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{1}{(cz+d)^2}. \tag{1.2.6}$$

By (1.2.5) we have

$$v = \frac{y}{|cz+d|^2}, \text{ therefore } \left|\frac{dw}{dz}\right| = \frac{v}{y}.$$

Thus

$$h(T(\gamma)) = \int_0^1 \frac{|\frac{dw}{dt}|dt}{v(t)} = \int_0^1 \frac{|\frac{dw}{dz}||\frac{dz}{dt}|dt}{v(t)} = \int_0^1 \frac{|\frac{dz}{dt}|dt}{y(t)} = h(\gamma).$$

The invariance of the hyperbolic distance follows from this immediately. □

1.3. Geodesics

THEOREM 1.3.1. *The geodesics in $\mathcal{H}$ are semicircles and the rays orthogonal to the real axis $\mathbb{R}$.*

PROOF. Let $z_1, z_2 \in \mathcal{H}$. First consider the case in which $z_1 = ia, z_2 = ib$ with $b > a$. For any piecewise differentiable curve $\gamma(t) = x(t) + iy(t)$ connecting ia and ib, we have

$$h(\gamma) = \int_0^1 \frac{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}}{y(t)} dt \geq \int_0^1 \frac{|\frac{dy}{dt}|dt}{y(t)} \geq \int_0^1 \frac{\frac{dy}{dt}dt}{y(t)} = \int_a^b \frac{dy}{y} = \ln\frac{b}{a},$$

but this is exactly the hyperbolic length of the segment of the imaginary axis connecting ia and ib. Therefore the geodesic connecting ia and ib is the segment of the imaginary axis connecting them.

Now consider the case of arbitrary points z_1 and z_2. Let L be the unique Euclidean semicircle or a straight line connecting them. Then by Exercise 4, there exists a transformation in $PSL(2,\mathbb{R})$ which maps L into the positive imaginary axis. This reduces the problem to the particular case studied above, so that by Theorem 1.2.4 we conclude that the geodesic between z_1 and z_2 is the segment of L joining them. □

Thus we have proved that any two points z and w in $\mathcal{H}$ can be joined by a unique geodesic, and the hyperbolic distance between them is equal to the hyperbolic length of the geodesic segment joining them; we denote the latter by $[z,w]$. This and the additivity of the integral (1.2.3) imply the following statement.

COROLLARY 1.3.2. *If z and w are two distinct points in $\mathcal{H}$, then*

$$\rho(z,w) = \rho(z,\xi) + \rho(\xi,w)$$

if and only if $\xi \in [z,w]$.

THEOREM 1.3.3. *Any isometry of $\mathcal{H}$, and in particular any transformation from $PSL(2,\mathbb{R})$, maps geodesics into geodesics.*

PROOF. The same argument as in the Euclidean case works here. □

The *cross–ratio* of distinct points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is defined by the following formula:

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}.$$

THEOREM 1.3.4. *Suppose $z, w \in \mathcal{H}$ are two distinct points, the geodesic joining z and w has endpoints $z*, w* \in \mathbb{R} \cup \{\infty\}$, and $z \in [z*, w]$. Then*

$$\rho(z,w) = \ln(w, z*; z, w*).$$

PROOF. Using Exercise 4, in $PSL(2,\mathbb{R})$ let us choose a transformation T which maps the geodesic joining z and w to the imaginary axis. By applying the transformations $z \mapsto kz\,(k > 0)$ and $z \mapsto -1/z$ if necessary, we may assume that $T(z*) = 0$, $T(w*) = \infty$ and $T(z) = i$. Then $T(w) = ri$ for some $r > 1$, and

$$\rho(T(z), T(w)) = \int_1^r \frac{dy}{y} = \ln r.$$

On the other hand, $(ri, 0; i, \infty) = r$, and the theorem follows from the invariance of the cross–ratio under Möbius transformations, a standard fact from complex analysis (which can be checked by a direct calculation). □

We will derive several explicit formulas for the hyperbolic distance involving the hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh z = \frac{\sinh x}{\cosh x}.$$

THEOREM 1.3.5. *For $z, w \in \mathcal{H}$, we have*

(a) $\rho(z,w) = \ln \frac{|z-\overline{w}|+|z-w|}{|z-\overline{w}|-|z-w|}$;
(b) $\cosh \rho(z,w) = 1 + \frac{|z-w|^2}{2\mathrm{Im(z)Im(w)}}$;
(c) $\sinh[\frac{1}{2}\rho(z,w)] = \frac{|z-w|}{2(\mathrm{Im(z)Im(w)})^{1/2}}$;
(d) $\cosh[\frac{1}{2}\rho(z,w)] = \frac{|z-\overline{w}|}{2(\mathrm{Im(z)Im(w)})^{1/2}}$;
(e) $\tanh[\frac{1}{2}\rho(z,w)] = |\frac{z-w}{z-\overline{w}}|$.

Proof. We will prove that (e) holds. By Theorem 1.2.4, the left–hand side is invariant under any transformation $T \in PSL(2,\mathbb{R})$. By Exercise 5, the right–hand side is also invariant under any $T \in PSL(2,\mathbb{R})$. Therefore if is sufficient to check the formula for the case when $z = i$, $w = ir$ $(r > 1)$. The right–hand side is equal to $(r-1)/(r+1)$. The left–hand side is equal to $\tanh[\frac{1}{2}\ln r]$. A simple calculation shows that these two expressions are equal. The other formulas are proved similarly. □

1.4. Isometries

We have seen that transformations in $PSL(2,\mathbb{R})$ are isometries of the hyperbolic plane $\mathcal{H}$ (Theorem 1.2.4). The next theorem identifies all isometries of $\mathcal{H}$ in terms of Möbius transformations and symmetry in the imaginary axis.

Theorem 1.4.1. *The group* $\mathrm{Isom}(\mathcal{H})$ *is generated by the Möbius transformations from* $PSL(2,\mathbb{R})$ *together with the transformation* $z \mapsto -\overline{z}$. *The group* $PSL(2,\mathbb{R})$ *is a subgroup of* $\mathrm{Isom}(\mathcal{H})$ *of index two.*

Proof. Let φ be any isometry of $\mathcal{H}$. By Theorem 1.3.3, φ maps geodesics into geodesics. Let I denote the positive imaginary axis. Then $\varphi(I)$ is a geodesic in $\mathcal{H}$, and according to Exercise 4, there exists an isometry $T \in PSL(2,\mathbb{R})$ that maps $\varphi(I)$ back to I. By applying the transformations $z \mapsto kz$ $(k > 0)$ and $z \mapsto -1/z$, we may assume that $g \circ \varphi$ fixes i and maps the rays (i,∞) and $(i,0)$ onto themselves. Hence, being an isometry, $g \circ \varphi$ fixes each point of I. The same (synthetic) argument as in the Euclidean case shows that

$$g \circ \varphi(z) = z \text{ or } -\overline{z}. \tag{1.4.1}$$

Let z_1 and z_2 be two fixed points on I. For any point z not on I, draw two hyperbolic circles centered at z_1 and z_2 and passing through z. These circles intersect in two points, z and $z' = -\overline{z}$, since the picture is symmetric with respect to the imaginary axis (note that a hyperbolic circle is a Euclidean circle in $\mathcal{H}$, but with a different center). Since these circles are mapped into themselves under the isometry $g \circ \varphi$, we conclude that $g \circ \varphi(z) = z$ or $g \circ \varphi(z) = -\overline{z}$. Since isometries are continuous (see Exercise 6), only one of the equations (1.4.1) holds for all $z \in \mathcal{H}$. If $g \circ \varphi(z) = z$, then $\varphi(z)$ is a

Möbius transformation of the form (1.2.4). If $g \circ \varphi(z) = -\overline{z}$, we have

$$\varphi(z) = \frac{a\overline{z} + b}{c\overline{z} + d} \text{ with } ad - bc = -1, \tag{1.4.2}$$

which proves the theorem. □

Thus we have characterized all the isometries of $\mathcal{H}$. The sign of the determinant of the corresponding matrix in (1.2.4) or (1.4.2) determines the *orientation* of an isometry. We will refer to transformations in $PSL(2, \mathbb{R})$ as *orientation–preserving* isometries and to transformations of the form (1.4.2) as *orientation–reversing* isometries.

Now we will study and classify these two types of isometries of the hyperbolic plane $\mathcal{H}$.

Orientation–preserving isometries. The classification of matrices in $SL(2, \mathbb{R})$ into hyperbolic, elliptic, and parabolic depended on the absolute value of their trace, and hence makes sense in $PSL(2, \mathbb{R})$ as well. A matrix $A \in SL(2, \mathbb{R})$ with trace t is called *hyperbolic* if $|t| > 2$, *elliptic* if $|t| < 2$, and *parabolic* if $|t| = 2$. Let

$$T(z) = \frac{az + b}{cz + d} \in PSL(2, \mathbb{R}).$$

The fixed points of T are found by solving the equation

$$z = \frac{az + b}{cz + d}, \quad \text{i.e.,} \quad cz^2 + (d - a)z - b = 0.$$

We obtain

$$w_1 = \frac{a - d + \sqrt{(a + d)^2 - 4}}{2c}, \quad w_2 = \frac{a - d - \sqrt{(a + d)^2 - 4}}{2c}.$$

We see that if T is hyperbolic, then it has two fixed points in $\mathbb{R} \cup \{\infty\}$, if T is parabolic, it has one fixed point in $\mathbb{R} \cup \{\infty\}$, and if T is elliptic, it has two complex conjugate fixed points, hence one fixed point in $\mathcal{H}$. A Möbius transformation T fixes ∞ if and only if $c = 0$, and hence it is in the form $z \mapsto az + b$ $(a, b \in \mathbb{R},\ a > 0)$. If $a = 1$, it is parabolic; if $a \neq 0$, it is hyperbolic and its second fixed point is $b/(1 - a)$. The fixed point w_i of T can be expressed in terms of the eigenvector $\begin{pmatrix} x_i \\ y_i \end{pmatrix}$ with eigenvalue λ_i, namely $w_i = x_i/y_i$. In terms of the eigenvalue λ_i, the derivative at the fixed point w_i can be written as itself:

$$T'(w_i) = \frac{1}{(cw_i + d)^2} = \frac{1}{\lambda_i^2}.$$

Definition. A fixed point w of a transformation $f : \mathcal{H} \to \mathcal{H}$ is called *attracting* if $|f'(w)| < 1$, and it is called *repelling* if $|f'(w)| > 1$.

Now we are ready to summarize what we know from linear algebra about different kinds of transformations in $PSL(2, \mathbb{R})$ and describe the action of Möbius transformations in $\mathcal{H}$ geometrically.

1. Hyperbolic case. A hyperbolic transformation $T \in PSL(2,\mathbb{R})$ has two fixed points in $\mathbb{R}\cup\{\infty\}$, one attracting, denoted by u, the other repelling, denoted by w. The geodesic in $\mathcal{H}$ connecting them is called the *axis* of T and is denoted by $C(T)$. By Theorem 1.3.3, T maps $C(T)$ onto itself, and $C(T)$ is the only geodesic with this property. Let λ be the eigenvalue of T with $|\lambda| > 1$. Then the matrix of T is conjugate to the diagonal matrix $\begin{pmatrix}\lambda & 0\\ 0 & 1/\lambda\end{pmatrix}$, which corresponds to the Möbius transformation

$$\Lambda(z) = \lambda^2 z, \tag{1.4.3}$$

i.e., there exists a transformation $S \in PSL(2,\mathbb{R})$ such that $STS^{-1} = \Lambda$. The conjugating transformation S maps the axis of T, oriented from u to w, to the positive imaginary axis I, oriented from 0 to ∞, which is the axis of Λ (cf. Exercises 4 and 9).

In order to see how a hyperbolic transformation T acts on $\mathcal{H}$, it is useful to look at the all its iterates T^n, $n \in \mathbb{Z}$. If $z \in C(T)$, then $T^n(z) \in C(T)$ and $T^n(z) \to w$ as $n \to \infty$, while $T^n(z) \to u$ as $n \to -\infty$. The curve $C(T)$ is the only geodesic which is mapped onto itself by T, but there are other T–invariant curves, also "connecting" u and w. For the standard hyperbolic transformation (1.4.3), the Euclidean rays in the upper half–plane issuing from the origin are obviously T–invariant. If we define the distance from a point z to a given geodesic L as $\inf_{v\in L}\rho(z,v)$, we see that the distance is measured over a geodesic passing through z and orthogonal to L (Exercise 7). Such rays have an important property: they are equidistant from the axis $C(\Lambda) = I$ (see Exercise 8), and hence are called *equidistants*. Under S^{-1} they are mapped onto equidistants for the transformation T, which are Euclidean circles passing through the points u and w (see Figure 1.4.1).

A useful notion in understanding how hyperbolic transformations act is that of *isometric circle*. Since $T'(z) = (cz+d)^{-2}$, the Euclidean lengths are multiplied by $|T'(z)| = |cz+d|^{-2}$. They are unaltered in magnitude if and only if $|cz+d| = 1$. If $c \neq 0$, then the locus of such points z is the circle

$$\left|z + \frac{d}{c}\right| = \frac{1}{|c|}$$

with center at $-d/c$ and radius $1/|c|$. The circle

$$I(T) = \{z \in \mathcal{H} \mid |cz+d| = 1\}$$

is called the *isometric circle* of the transformation T. Since its center $-d/c$ lies in $\mathbb{R}$, we immediately see that isometric circles are geodesics in $\mathcal{H}$. Further, $T(I(T))$ is a circle of the same radius, $T(I(T)) = I(T^{-1})$, and the transformation maps the outside of $I(T)$ onto the inside of $I(T^{-1})$ and vice versa (see Figure 1.4.1 and Exercise 10.

If $c = 0$, then there is no circle with the isometric property: all Euclidean lengths are altered.

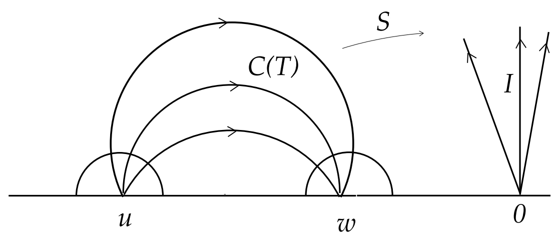

FIGURE 1.4.1. Hyperbolic transformations

2. Parabolic case. A parabolic transformation $T \in PSL(2,\mathbb{R})$ has one fixed point $p \in \mathbb{R} \cup \{\infty\}$. The transformation T has one eigenvalue $\lambda = \pm 1$ and is conjugate to the transformation $P(z) = z + b$ for some $b \in \mathbb{R}$, i.e., there exists a transformation $S \in PSL(2,\mathbb{R})$ such that $P = STS^{-1}$. The transformation P is an Euclidean translation, and hence it leaves all horizontal lines invariant. Horizontal lines are called *horocycles* for the transformation P. Under the map S^{-1} they are sent to invariant curves (horocycles) for the transformation T. Horocycles for T are Euclidean circles tangent to the real line at the parabolic fixed point p (see Figure 1.4.2 and Exercise 3.1.2).

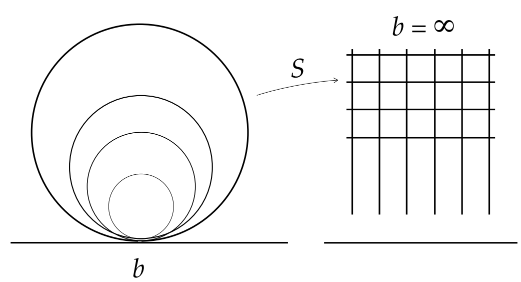

FIGURE 1.4.2. Parabolic transformations

If $c \neq 0$, then the isometric circles for T and T^{-1} are tangent to each other (see Exercise 11). If $c = 0$, then there is no unique circle with the isometric property: in this case T is an Euclidean translation, all Euclidean lengths are unaltered.

3. Elliptic case. An elliptic transformation $T \in PSL(2,\mathbb{R})$ has a unique fixed point $e \in \mathcal{H}$. It has the eigenvalues $\lambda = \cos\varphi + i\sin\varphi$ and $\overline{\lambda} = \cos\varphi - i\sin\varphi$, and it is easier to describe its simplest form in the *unit disc model* of hyperbolic geometry: $\mathcal{U} = \{z \in \mathbb{C} \mid |z| < 1\}$. The map

$$f(z) = \frac{zi + 1}{z + i} \tag{1.4.4}$$

is a homeomorphism of $\mathcal{H}$ onto $\mathcal{U}$. The distance in $\mathcal{U}$ is induced by means of the hyperbolic distance in $\mathcal{H}$:

$$\rho(z,w) = \rho(f^{-1}z, f^{-1}w) \ (z,w \in \mathcal{U}).$$

The readily verified formula

$$\frac{2|f'(z)|}{1-|f(z)|^2} = \frac{1}{\mathrm{Im(z)}}$$

implies that this distance in $\mathcal{U}$ is derived from the metric

$$ds = \frac{2|dz|}{1-|z|^2}.$$

Geodesics in the unit disc model are circular arcs and diameters orthogonal to the *principle circle* $\Sigma = \{z \in \mathbb{C} \mid |z| = 1\}$, the *Euclidean boundary* of $\mathcal{U}$. Isometries of $\mathcal{U}$ are the conjugates of isometries of $\mathcal{H}$, i.e., we can write

$$S = f \circ T \circ f^{-1} \ (T \in PSL(2,\mathbb{R})).$$

Exercise 13 shows that orientation–preserving isometries of $\mathcal{U}$ are of the form

$$z \mapsto \frac{az+\overline{c}}{cz+\overline{a}} \ (a,c \in \mathbb{C}, a\overline{a} - c\overline{c} = 1),$$

and the transformation corresponding to the standard reflection $R(z) = -\overline{z}$ is also the reflection of $\mathcal{U}$ in the vertical diameter.

Let us return to our elliptic transformation $T \in PSL(2,\mathbb{R})$ that fixes $e \in \mathcal{H}$. Conjugating T by f, we obtain an elliptic transformation of the unit disc $\mathcal{U}$. Using an additional conjugation by an orientation–preserving isometry of $\mathcal{U}$ if necessary (see Exercise 3.2.3), we bring the fixed point to 0, and hence bring T to the form $z \mapsto e^{2i\varphi}z$. In other words, an elliptic transformation with eigenvalues $e^{i\varphi}$ and $e^{-i\varphi}$ is conjugate to a rotation by 2φ.

Example 1 . Let $z \mapsto -1/z$ be the elliptic transformation given by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Its fixed point in $\mathcal{H}$ is i. It is a transformation of order 2 since the identity in $PSL(2,\mathbb{R})$ is $\{1_2, -1_2\}$, and hence is a half–turn. In the unit disc model, its matrix is conjugate to the matrix $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

Orientation–reversing isometries. The simplest orientation–reversing isometry of $\mathcal{H}$ is the transformation $R(z) = -\overline{z}$, which is the reflection in the imaginary axis I, and hence it fixes I pointwise. It is also a hyperbolic reflection in I, i.e., if for each point z we draw a geodesic through z, orthogonally to I and intersecting I at a point z_0, then $R(z) = z'$ is on the same geodesic and $\rho(z', z_0) = \rho(z, z_0)$. Let L be any geodesic in $\mathcal{H}$ and $T \in PSL(2,\mathbb{R})$ be any Möbius transformation. Then the transformation

$$TRT^{-1} \tag{1.4.5}$$

fixes the geodesic $L = T(I)$ pointwise and therefore may be regarded as a "reflection in the geodesic L". In fact, it is the well–known geometrical transformation called *inversion in a circle.*

DEFINITION. Let Q be a circle in $\mathbb{R}^2$ with center K and radius r. Given any point $P \neq K$ in $\mathbb{R}^2$, a point P_1 is called *inverse* to P if

(a) P_1 lies on the ray from K to P,
(b) $|KP_1| \cdot |KP| = r^2$.

The relationship is reciprocal: if P_1 is inverse to P, then P is inverse to P_1. We say that P and P_1 are *inverse with respect to* Q. Obviously, inversion fixes all points in the circle Q. Inversion may be described by a geometric construction (see Exercise 15). We will derive a formula for it. Let P, P_1 and K be the points z, z_1, and k in $\mathbb{C}$. Then the definition can be rewritten as

$$|(z_1 - k)(z - k)| = r^2, \quad \arg(z_1 - k) = \arg(z - k).$$

Since $\arg(z - k) = -\arg(\overline{z} - \overline{k})$, both equations are satisfied if and only if

$$(z_1 - k)(\overline{z} - \overline{k}) = r^2. \tag{1.4.6}$$

This gives us the following formula for the inversion in a circle:

$$z_1 = \frac{k\overline{z} + r^2 - |k|^2}{\overline{z} - \overline{k}}. \tag{1.4.7}$$

Now we are able to prove a theorem for isometries of the hyperbolic plane similar to a result in Euclidean geometry.

THEOREM 1.4.2. *Every isometry of $\mathcal{H}$ is a product of not more than three reflections in geodesics in $\mathcal{H}$.*

PROOF. By Theorem 1.4.1 it suffices to show that each transformation from the group $PSL(2, \mathbb{R})$ is a product of two reflections. Let

$$T(z) = \frac{az + b}{cz + d}.$$

First consider the case for which $c \neq 0$. Then both T and T^{-1} have well–defined isometric circles (see Exercise 11). They have the same radius $1/|c|$ and their centers are on the real axis at $-d/c$ and a/c, respectively. We will show that $T = R \circ R_{I(T)}$, where $R_{I(T)}$ is the reflection in the isometric circle $I(T)$, or inversion, and R is the reflection in the vertical geodesic passing through the midpoint of the interval $[-d/c, a/c]$. To do this, we use formula (1.4.6) for inversion:

$$R_{I(T)}(z) = \frac{-\frac{d}{c}\overline{z} + \frac{1}{c^2} - \frac{d^2}{c^2}}{\overline{z} + \frac{d}{c}} = \frac{-d(\overline{z} + \frac{d}{c}) + \frac{1}{c}}{c\overline{z} + d}.$$

The reflection in the line $x = (a - d)/2c$ is given by the formula

$$R(z) = -\overline{z} + 2\frac{a - d}{2c}.$$

Combining the two, we obtain

$$R \circ R_{I(T)} = \frac{az+b}{cz+d}.$$

Now if $c = 0$, the transformation T may be either parabolic $z \mapsto z+b$ or hyperbolic $z \mapsto \lambda^2 z + b$, each fixing ∞. In the first case, the theorem follows from the Euclidean result for translations. For $T(z) = \lambda^2 z + b$, it is easy to see that the reflections should be in circles of radii 1 and λ centered at the second fixed point. $\square$

1.5. Hyperbolic area and the Gauss–Bonnet formula

Let T be a Möbius transformation. The *differential* of T, denoted by DT, at a point z is the linear map that takes the tangent space $T_z\mathcal{H}$ onto $T_{T(z)}\mathcal{H}$ and is defined by the 2×2 matrix

$$DT = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

THEOREM 1.5.1. *Let $T \in PSL(2,\mathbb{R})$. Then DT preserves the norm in the tangent space at each point.*

PROOF. For $\zeta \in T_z\mathcal{H}$, we have $DT(\zeta) = T'(z)\zeta$ by Exercise 21. Since

$$|T'(z)| = \frac{\mathrm{Im(T(z))}}{\mathrm{Im(z)}} = \frac{1}{|cz+d|^2},$$

we can write

$$\|DT(\zeta)\| = \frac{|DT(\zeta)|}{\mathrm{Im(T(z))}} = \frac{|T'(z)||\zeta|}{\mathrm{Im(T(z))}} = \frac{|\zeta|}{\mathrm{Im(z)}} = \|\zeta\|.$$

$\square$

COROLLARY 1.5.2. *Any transformation in $PSL(2,\mathbb{R})$ is conformal, i.e., it preserves angles.*

PROOF. It is easy to prove the *polarization identity*, which asserts that for any $\zeta_1, \zeta_2 \in T_z\mathcal{H}$ we have

$$\langle \zeta_1, \zeta_2 \rangle = \frac{1}{2}(\|\zeta_1\|^2 + \|\zeta_2\|^2 - \|\zeta_1 - \zeta_2\|^2);$$

this identity implies that the inner product and hence the absolute value of the angle between tangent vectors is also preserved. Since Möbius transformations preserve orientation, the corollary follows. $\square$

Let $A \subset \mathcal{H}$. We define the *hyperbolic area* of A by the formula

$$\mu(A) = \int_A \frac{dxdy}{y^2}, \tag{1.5.1}$$

provided this integral exists.

THEOREM 1.5.3. *Hyperbolic area is invariant under all Möbius transformations $T \in PSL(2,\mathbb{R})$, i.e., if $\mu(A)$ exists, then $\mu(A) = \mu(T(A))$.*

PROOF. It follows immediately from the preservation of Riemannian metric (Theorem 1.5.1). Here is a direct calculation as well. When we performed the change of variables $w = T(z)$ in the line integral of Theorem 1.2.4, the coefficient $|T'(z)|$ appeared (it is the coefficient responsible for the change of Euclidean lengths). If we carry out the same change of variables in the plane integral, the Jacobian of this map will appear, since it is responsible for the change of the Euclidean areas. Let $z = x+iy$, and $w = T(z) = u+iv$.

The Jacobian is the determinant of the differential map DT and is customarily denoted by $\partial(u,v)/\partial(x,y)$. Thus

$$\frac{\partial(u,v)}{\partial(x,y)} := \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial x}\Big)^2 = |T'(z)|^2 = \frac{1}{|cz+d|^4}. \quad (1.5.2)$$

We use this expression to compute the integral

$$\begin{aligned}\mu(T(A)) &= \int_{T(A)} \frac{dudv}{v^2} = \int_A \frac{\partial(u,v)}{\partial(x,y)} \frac{dxdy}{v^2} \\ &= \int_A \frac{1}{|cz+d|^4} \frac{|cz+d|^4}{y^2} dxdy = \mu(A),\end{aligned}$$

as claimed. □

A *hyperbolic triangle* is a figure bounded by three segments of geodesics. The intersection points of these geodesics are called the *vertices* of the triangle. We allow vertices to belong to $\mathbb{R} \cup \{\infty\}$. There are 4 types of hyperbolic triangles, depending on whether $0, 1, 2$, or 3 vertices belong to $\mathbb{R} \cup \{\infty\}$ (see Figure 1.5.1).

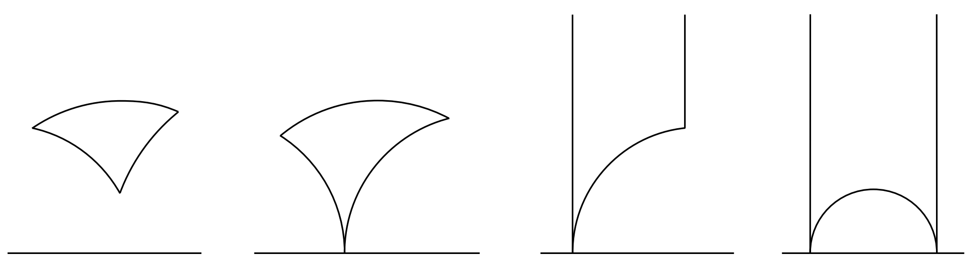

FIGURE 1.5.1. Hyperbolic triangles

The Gauss–Bonnet formula shows that the hyperbolic area of a hyperbolic triangle depends only on its angles.

THEOREM 1.5.4 (Gauss-Bonnet). *Let Δ be a hyperbolic triangle with angles α, β, and γ. Then $\mu(\Delta) = \pi - \alpha - \beta - \gamma$.*

PROOF. First we consider the case in which one of the vertices of the triangle belongs to $\mathbb{R} \cup \{\infty\}$. Since transformations from $PSL(2,\mathbb{R})$ do not alter the area and the angles of a triangle, we may apply the transformation from $PSL(2,\mathbb{R})$ which maps this vertex to ∞ and the base to a segment of the unit circle (as in Figure 1.5.2), and prove the formula in this case.

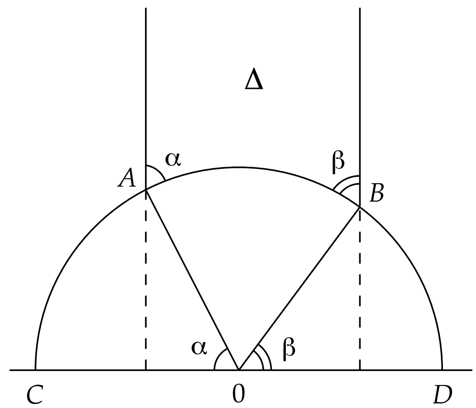

FIGURE 1.5.2

The angle at infinity is equal to 0, and let us assume that the other two angles are equal to α and β. Since the angle measure in the hyperbolic plane coincides with the Euclidean angle measure, the angles A0C and B0D are equal to α and β, respectively, as angles with mutually perpendicular sides. Assume the vertical geodesics are the lines $x = a$ and $x = b$. Then

$$\mu(\Delta) = \int_\Delta \frac{dxdy}{y^2} = \int_a^b dx \int_{\sqrt{1-x^2}}^\infty \frac{dy}{y^2} = \int_a^b \frac{dx}{\sqrt{1-x^2}}.$$

The substitution $x = \cos\theta$ $(0 \leq \theta \leq \pi)$ gives

$$\mu(\Delta) = \int_{\pi-\alpha}^{\beta} \frac{-\sin\theta d\theta}{\sin\theta} = \pi - \alpha - \beta.$$

For the case in which Δ has no vertices at infinity, we continue the geodesic connecting the vertices A and B, and suppose that it intersects the real axis at the point D (if one side of Δ is a vertical geodesic, then we label its vertices A and B), and draw a geodesic from C to D. Then we obtain the situation depicted in Figure 3.2.1.

We denote the triangle ADC by Δ_1 and the triangle CBD by Δ_2. Our formula has already been proved for triangles such as Δ_1 and Δ_2, since the vertex D is at infinity. Now we can write

$$\begin{aligned}\mu(\Delta) = \mu(\Delta_1) - \mu(\Delta_2) &= (\pi - \alpha - \gamma - \theta) - (\pi - \theta - \pi + \beta)\\ &= \pi - \alpha - \beta - \gamma,\end{aligned}$$

as claimed. □

Theorem 1.5.4 asserts that the area of a triangle depends only on its angles, and is equal to the quantity $\pi - \alpha - \beta - \gamma$, which is called the *angular defect*. Since the area of a nondegenerate triangle is positive, the

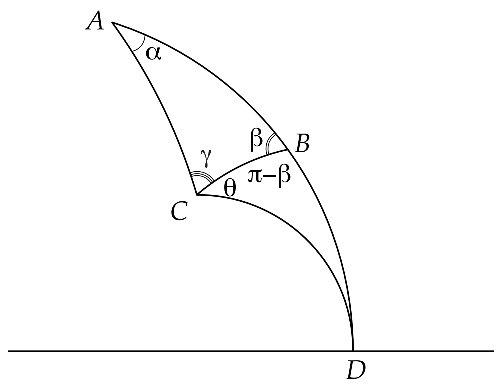

FIGURE 1.5.3

angular defect is positive, and therefore, in hyperbolic geometry the sum of angles of any triangle is less than π. We will also see that there are no similar triangles in hyperbolic geometry (except isometric ones).

THEOREM 1.5.5. *If two triangles have the same angles, then there is an isometry mapping one triangle into the other.*

PROOF. If necessary, we perform the reflection $z \mapsto -\overline{z}$, so that the respective angles of the triangles ABC and $A'B'C'$ (in the clockwise direction) are equal. Then we apply a hyperbolic transformation mapping A to A' (Exercise 3.2.3), and an elliptic transformation mapping the side AB onto the side $A'B'$. Since the angles CAB and $C'A'B'$ are equal, the side AC will be mapped onto the side $A'C'$. We must prove that B is then mapped to B' and C to C'. Assume B' is mapped inside the geodesic segment AB. If we had $C' \in [A, C]$, the areas of triangles ABC and $A'B'C'$ would not be equal, which contradicts Theorem 1.5.4. Therefore C must belong to the side $A'C'$, and hence the sides BC and $B'C'$ intersect at a point X (see Fig. 1.5.4); thus we obtain the triangle $B'XB$. Its angles are β and $\pi - \beta$ since the angles at the vertices B and B' of our original triangles are equal (to β). We see that, in contradiction with Theorem 1.5.4, the sum of the angles of the triangle $B'XB$ is at least π. $\square$

1.6. Hyperbolic trigonometry

Let us consider a general hyperbolic triangle with sides of hyperbolic length a, b, c and opposite angles α, β, γ. We assume that α, β, and γ are positive (so a, b, and c are finite) and prove the following results.

THEOREM 1.6.1. *(i) The Sine Rule:* $\frac{\sinh\, a}{\sin\alpha} = \frac{\sinh\, b}{\sin\beta} = \frac{\sinh\, c}{\sin\gamma}$.

(ii) The Cosine Rule I: $\cosh\, c = \cosh\, a \cosh\, b - \sinh\, a \sinh\, b \cos\gamma$.

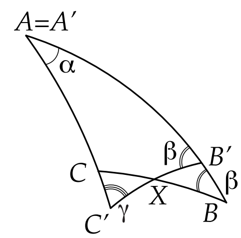

FIGURE 1.5.4

(iii) The Cosine Rule II: $\cosh\ c = \frac{\cos\alpha\cos\beta+\cos\gamma}{\sin\alpha\sin\beta}$.

REMARK. Note the existence of Cosine Rule II. This has no analogue in Euclidean geometry: in hyperbolic geometry it implies that *if two triangles have the same angles, then there is an isometry mapping one triangle onto the other.*

PROOF OF (ii). Let us denote the vertices opposite the sides a, b, c by v_a, v_b, v_c respectively. We shall use the model $\mathcal{U}$ and may assume that $v_c = 0$ and $\operatorname{Im} v_a = 0$, Re $v_a > 0$ (see Figure 1.6.1). By Exercise 19(iv) we have

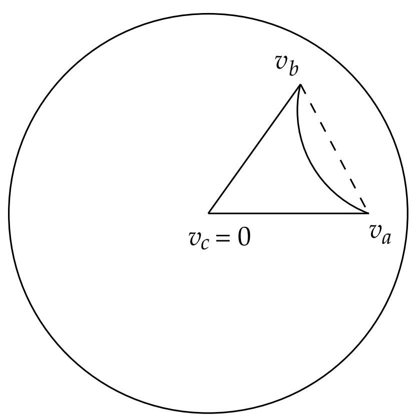

FIGURE 1.6.1

$$v_a = \tanh\frac{1}{2}\rho(0, v_a) = \tanh(\frac{1}{2}b), \tag{1.6.1}$$

and similarly,

$$v_b = e^{i\gamma}\tanh(\frac{1}{2}a), \tag{1.6.2}$$

We have $c = \rho(v_a, v_b)$, and from Exercise 19(iii)

$$\cosh\ c = \sinh^2[\frac{1}{2}\rho(v_a, v_b)] + 1 = \frac{2|v_a - v_b|^2}{(1-|v_a|^2)(1-|v_b|^2)} + 1. \tag{1.6.3}$$

The right-hand side of expression (1.6.3) is equal to $\cosh a \cosh b - \sinh a \sinh b \cos\gamma$ by Exercise 22, and hence (ii) follows. □

Proof of (i). Using (ii) we obtain

$$\left(\frac{\sinh c}{\sin\gamma}\right)^2 = \frac{\sinh^2 c}{1 - \left(\frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b}\right)^2}. \tag{1.6.4}$$

The Sine Rule will be valid if we prove that the expression on the right-hand side of (1.6.4) is symmetric in a, b, and c. This follows from the symmetry of

$$(\sinh a \sinh b)^2 - (\cosh a \cosh b - \cosh c)^2$$

which is obtained by a direct calculation. □

Proof of (iii). Let us write A for $\cosh a$, B for $\cosh b$, and C for $\cosh c$. The Cosine Rule I yields

$$\cos\gamma = \frac{(AB - C)}{(A^2-1)^{\frac{1}{2}}(B^2-1)^{\frac{1}{2}}}$$

and so

$$\sin^2\gamma = \frac{D}{(A^2-1)(B^2-1)}$$

where $D = 1 + 2ABC - (A^2 + B^2 + C^2)$ is symmetric in A, B, and C. The expression for $\sin^2\gamma$ shows that $D \geq 0$. Using analogous expressions for $\cos\alpha$, $\sin\alpha$, $\cos\beta$, and $\sin\beta$ we observe that if we multiply both the numerator and denominator of

$$\frac{\cos\alpha\ \cos\beta + \cos\gamma}{\sin\alpha\ \sin\beta}$$

by the positive value of

$$(A^2-1)^{\frac{1}{2}}(B^2-1)^{\frac{1}{2}}(C^2-1)^{\frac{1}{2}}$$

we obtain

$$\frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta} = \frac{[(BC-A)(CA-B) + (AB-C)(C^2-1)]}{D} = C$$

□

Theorem 1.6.2. (Pythagorian Theorem) If $\gamma = \frac{\pi}{2}$ we have $\cosh c = \cosh a \cosh b$.

Proof. Immediate from the Cosine Rule I. □

Exercises

1. Prove that the metric in the Poincaré disc model is given by

$$ds^2 = \frac{4(d\eta_1^2 + d\eta_2^2)}{(1-(\eta_1^2+\eta_2^2))^2}.$$

2. Prove that the metric in the upper half–plane model is given by

$$ds^2 = \frac{d\eta_1^2 + d\eta_3^2}{\eta_3^2}.$$

3. Prove that if $z \neq w$, then $\rho(z, w) > 0$.

4. Let L be a semicircle or a straight line orthogonal to the real axis which meets the real axis at a point α. Prove that the transformation

$$T(z) = -(z-\alpha)^{-1} + \beta \in PSL(2, \mathbb{R}),$$

for an appropriate value of β, maps L to the positive imaginary axis.

5. Prove that for $z, w \in \mathcal{H}$ and $T \in PSL(2, \mathbb{R})$, we have

$$|T(z) - T(w)| = |z-w||T'(z)T'(w)|^{1/2}.$$

6. Prove that isometies are continuous maps.

7. (a) Prove that there is a unique geodesic through a point z orthogonal to a given geodesic L.

(b)* Give a geometric construction of this geodesic.

(c) Prove that for $z \notin L$, the greatest lower bound $\inf_{v \in L} \rho(z, v)$ is achieved on the geodesic described in (a).

8. Prove that the rays in $\mathcal{H}$ issuing from the origin are equidistant from the positive imaginary axis I.

9. Let $A \in PSL(2, \mathbb{R})$ be a hyperbolic transformation, and suppose that $B = SAS^{-1}$ ($B \in PSL(2, \mathbb{R})$) is its conjugate. Prove that B is also hyperbolic and find the relation between their axes $C(A)$ and $C(B)$.

10. Prove that isometric circles $I(T)$ and $I(T^{-1})$ have the same radius, and that the image of $I(T)$ under the transformation T is $I(T^{-1})$.

11. Prove that

(a) T is hyperbolic if and only if $I(T)$ and $I(T^{-1})$ do not intersect;
(b) T is elliptic if and only if $I(T)$ and $I(T^{-1})$ intersect;
(c) T is parabolic if and only if $I(T)$ and $I(T^{-1})$ are tangential.

12. Prove that the horocycles for a parabolic transformation with a fixed point $p \in \mathbb{R}$ are Euclidean circles tangent to the real line at p.

13. Show that orientation–preserving isometries of $\mathcal{U}$ are of the form

$$z \mapsto \frac{az+\overline{c}}{cz+\overline{a}} \qquad (a, c \in \mathbb{C}, a\overline{a} - c\overline{c} = 1).$$

14. Prove that for any two distinct points $z_1, z_2 \in \mathcal{H}$ there exists a transformation $T \in PSL(2, \mathbb{R})$ such that $T(z_1) = z_2$.

15. Give a geometric construction of the inversion in a given circle Q in the Euclidean plane $\mathbb{R}^2$.

16. Prove that the transformation (1.4.5) is an inversion in the circle corresponding to the geodesic L.

17. Prove that two hyperbolic transformations in $PSL(2, \mathbb{R})$ commute if and only if their axes coincide.

18. Let $A \in PSL(2, \mathbb{R})$ be hyperbolic and $B \in PSL(2, \mathbb{R})$ be an elliptic transformation different from the identity. Prove that $AB \neq BA$.

19. Use the map f (1.4.4) to derive the formulae for the hyperbolic distance in the unit disc model similar to those in Theorem 1.3.5, for $z, w \in \mathcal{U}$:

(i) $\rho(z, w) \in \ln \frac{|1-z\bar{w}|+|z-w|}{|1-z\bar{w}|-|z-w|}$,

(ii) $\cosh^2[\frac{1}{2}\rho(z, w)] = \frac{|1-z\bar{w}|^2}{(1-|z|^2)(1-|w|^2)}$,

(iii) $\sinh^2[\frac{1}{2}\rho(z, w)] = \frac{|z-w|^2}{(1-|z|^2)(1-|w|^2)}$,

(iv) $\tanh[\frac{1}{2}\rho(z, w)] = |\frac{z-w}{1-z\bar{w}}|$.

20. Justify the calculations in (1.5.2) by checking that for the Möbius transformation

$$w = T(z) = \frac{az+b}{cz+d} \quad \text{with} \quad z = x + iy,\ w = u + iv$$

we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

(these are the classical *Cauchy–Riemann equations*) and

$$T'(z) = \frac{dw}{dz} = \frac{1}{2}\Big(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\Big) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x};$$

(*Hint*: express x and y in terms of z and $\overline{z}$ and use the Cauchy–Riemann equations.)

21. If we identify the tangent space $T_z\mathcal{H} \approx \mathbb{R}^2$ with the complex plane $\mathbb{C}$ by means of the map

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \xi + i\eta = \zeta,$$

then $DT(\zeta) = T'(z)\zeta$, where in the left–hand side we have a linear transformation of $T_z\mathcal{H} \approx \mathbb{R}^2$, and in the right–hand side, the multiplication of two complex numbers.

22. Show that the right-hand side of expression (1.6.3) is equal to
$\cosh a \cosh b - \sinh a \sinh b \cos\gamma$.