Problems from the midterm: solutions and comments

Problem 1.1. *Consider the torus T obtained from a regular hexagon by identifying pairs of opposite sides by translations. How many fixed points can an isometry of T have?*

Solution: The first step is to identify the isometries of T. The group of isometries is generated by translations of the plane together with the dihedral group D_6, which comprises rotations and reflections and can be interpreted as the group of isometries of the hexagon. Upon composition of these generators, we obtain other rotations and reflections as well as glide reflections along certain lines. In fact, any line forming an angle of $\frac{k\pi}{6}$ with the horizontal will do. An important point to keep in mind is that any rotation is conjugate via a translation to a rotation around the centre of the hexagon and hence has the same number of fixed points.

The next step is to count fixed points of these isometries. A non-trivial translation has no fixed points, and a reflection has infinitely many. Because a glide reflections preserves the family of lines parallel to its axis and acts as a translation on any such line, the existence of a single fixed point for such an isometry implies the existence of an entire line of fixed points. It remains to consider rotations around the center of the hexagon.

Rotation around the centre of the hexagon will not fix any other point in the interior, and so we need only consider the edges and vertices. The six vertices of the hexagon are identified in two equivalence classes, each corresponding to a different point on the torus. Rotation by $\pm\frac{\pi}{3}$ interchanges these two vertices and permutes the edges, and so has only one fixed point, the centre. Rotation by $\pm\frac{2\pi}{3}$ leaves these two vertices fixed and permutes the edges, and so has three fixed points. Finally, rotation by π exchanges the vertices but acts as the flip on each of the edges and in particular fixes the midpoint of each edge, and so has four fixed points.

Thus the possible numbers of fixed points are 0, 1, 3, 4, and ∞. $\square$

Comment: A completely rigorous discussion (which was not required for the exam problem) would require proving that the list above exhausts all isometries. This can be achieved by looking at simple (non self-intersecting) *closed geodesics*. There are three parallel families forming the angles $\frac{\pi}{3}$ with each other. Isometry moves any such family to another family. Composing if necessary a given orientation preserving isometry I with a rotation by a multiple of $\frac{\pi}{3}$ one produces an isometry which preserves each family. Since an isometry is uniquely determined by images of three points (as in the whole plane) this composition is a translation. Thus the isometry I is either translation itself or a product of translation and rotation, hence a rotation by $\frac{k\pi}{3}$. For an orientation reversing isometry take composition with a fixed reflection which gives one of the above orientation preserving isometries.

Problem 1.2. *A* regular map of type (p, q) *on a surface is a map such that every face has* p *sides and every vertex has* p *edges attached to it. Notice that in both cases an edge may be counted twice if, correspondingly, the same face lies on both sides of the edge, or if the edge is a loop.*

Find for what valus of p *and* q *there exists a regular map of type* (p, q) *on the torus.*

Solution: The corresponding problem for the sphere led us to the five Platonic solids, and we adopt a similar approach here. Each face has p sides, so $pF = 2E$, and similarly $qV = 2E$. Hence we have

$$\chi = V - E + F = (\frac{2}{q} - 1 + \frac{2}{p})E$$

Since the Euler characteristic of the torus is zero, this leads us to the necessary condition

$$\frac{1}{q} + \frac{1}{p} = \frac{1}{2}$$

Thus both p and q are greater than 2 and at least one must be smaller than 5 since $\frac{1}{5} + \frac{1}{5} < \frac{1}{2}$. Thus the only positive integer solutions to this equation are $(3, 6)$, $(4, 4)$, and $(6, 3)$. These correspond to the three regular tesselations of the plane, by triangles, squares, and hexagons, respectively; each of these can be 'rolled up' onto the torus (in infinitely many different ways) to demonstrate the existence of (infinitely many) (p, q)-regular maps of a given type. $\square$

Comment: The maps with the smallest number of elements are

- type $(4, 4)$: the square with opposite sides identified, i.e. the first standard model for the torus:
- type $(6, 3)$: the (regular) hexagon with opposite sides identified, i.e. the second standard model for the torus:
- type $(3, 6)$: a parallelogram with opposite sides identified, and diagonal added.

Notice the difference with the sphere case where there is only one model for each of the five admissible types.

Question 2.1. *Describe all isometries of the standard torus of revolution ("the bagel") listing orientation preserving and orientation reversing separately. Is it true that every isometry extends to an isometry of* $\mathbb{R}^3$?

Solution: The orientation preserving isometries are given by (i) rotations around the z-axis by any angle, along with (ii) rotation around any line through the origin in the xy-plane by the specific angle π.

The orientation reversing isometries are given by (iii) reflections through any vertical plane containing the z-axis, along with (iv) reflection through the xy-plane composed with a rotation around z axis.

All isometries extend to isometries of $\mathbb{R}^3$. □

Comment: We note first two common mistakes. First, the torus of revolution is not isometric to the flat torus obtained from a planar model on a square, and so it will do us no good to consider isometries of the latter.

Second, if we take the z-axis to be the axis of rotation, so that the torus is given as the set of solutions to

$$(\sqrt{x^2+y^2}-2)^2+z^2=1$$

then the map of the surface obtained by rotating the vertically-oriented circles is not an isometry.

A proof that the list is complete (not required) can be obtained as follows: notice that the long and short horizontal circles (in the intersection with the xy-plane) can be characterized as the only simple (non self-intersecting) *isolated* closed geodesics. Hence every isometry preserves each of those circles. On the circle the isometry is either a rotation or a reflection. Isometries preserve angles and take geodesics into geodesics. Hence vertical circles are mapped into vertical circles preserving distance along those. Thus if the big horizontal circle rotates each vertical circle either just rotates with it (rotation; case (i)) or rotates and reflects in the horizontal plane (case (iv)). If the big horizontal circle is reflected then two vertical circles are fixed and hence the isometry either fixes each point of those circles (case (iii)) or is the reflection along the horizontal axis (case(ii)).