

MASS-07; Geometry MWF 11:15am - 12:05pm.

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## **SURFACES: (ALMOST) EVERYTHING YOU WANTED TO KNOW ABOUT THEM**

Surfaces form one of the most basic and pervasive mathematical images; they appear visually as shapes embedded into the ordinary space, and also as configuration or phase spaces of simple mechanical systems, and as natural domains of definitions of functions defined by most basic algebraic formulas.

Attempts to describe these objects systematically and classify them lead to the development of the central concepts of geometry, topology, complex analysis, and even algebra. In this course we will develop some of those concepts and apply them to the study and classification of surfaces.

The word “surface” below normally means a compact surface without boundary.

*PLAN OF THE COURSE. Very approximately each section with an Arabic numeral corresponds to one lecture.*

### I. PRELUDE

1. Name your favorite surface.

### II. INTRODUCTION

2. Surfaces as shapes in Euclidean space. Various ways and modes of description and difficulties associated with them.

3. Internal geometry of embedded surfaces; angles, length of curves, shortest curves (geodesics), isometries. Examples when isometries cannot be effected by a motion of the ambient space.

4. Examples of non-orientable surfaces in this order: Mobius strip (embedded), Klein bottle (Immersed with a self-intersection), projective plane (how to visualize?)

5. Embedding to higher-dimensional spaces: flat torus, Klein bottle, projective plane.

### III. BEGINNING OF SYSTEMATIC THEORY: TRIANGULATIONS AND COMBINATORIAL STRUCTURE

6. Surfaces as combinatorial and “stretchable” objects. Maps (as in “geographic map”) and triangulations. Euler theorem for maps on the plane and on the sphere.

7. Handles, Mobius caps and inverted handles. Equivalence of attaching a pair of Mobius caps and an inverted handle; also of a triple of Mobius caps and a Mobius cap and a handle. The standard list of surfaces: spheres with handles and spheres with one or two Mobius caps and handles.

8. Euler characteristic of a map. Generalized Euler theorem for maps on arbitrary surfaces. First definition of Euler characteristic for a surface.

9. Orientability. Sphere with handles is orientable. Sphere with Mobius caps (and handles) is non-orientable.

10. Reduction of any surface to one for the standard list by reducing to a polygon with pairs of identified sides and re-glueing.

11. Characterization of surfaces by Euler characteristic and orientability. Definition of Betti numbers. The second guise of the Euler characteristic: alternating sum of Betti numbers.

### IV. SMOOTH STRUCTURE, FUNCTIONS AND DIFFERENTIAL EQUATIONS

12. How one gets two coordinates locally on a surface embedded into Euclidean space: projections to coordinate planes, to tangent planes, stereographic projection, application of Implicit Function Theorem.

13. How local? Examples: spherical and elliptical coordinates, coordinates on surfaces of revolution. Surfaces defined by equations in three and more variables. Critical and non-critical values. Connection with Implicit Function Theorem.

14. Definition of smooth structure through local coordinates and compatibility: a surface is a two-dimensional differentiable manifold. Coordinate atlases on the surfaces from the standard list from N7.

15. Functions on surfaces. Regular and critical points. Independence of choice of coordinates. Non-degenerate critical points: the Hessian of second derivatives has non-zero determinant. Classification of non-degenerate critical points into minima, maxima, and saddles. Morse Lemma.

16. Morse functions in the broad sense: all critical point are non-degenerate. Examples of Morse functions: the height function on a

typical embedded surface, the altitude function of the “landscape” associated with a triangulation.

17. Index of an isolated critical point. Calculation of index for a maximum, minimum and a (multiple) saddle. Construction of a function with three critical points on any standard surface.

18. The third guise of Euler characteristic: the sum of indices of critical points for a function.

19. Vector fields on surfaces. Vector field as a coordinate-free form of a first order ODE. Zeroes of a vector field. Index of an isolated zero. Classification of non-degenerate zeroes.

20. The fourth guise of Euler characteristic: the sum of indices of critical points for a vector field.

## V. TOPOLOGY OF SURFACES

21. Jordan separation theorem for simple closed curves on the plane and on the sphere. Proof for smooth and polygonal curves. Genus of the surface as the maximal number of non-intersecting smooth curves which do not divide the surface.

22. Jordan and Schoenflies separation theorems for arbitrary simple closed curves. Proof by approximation.

23. Homeomorphism. Existence of a triangulation on any compact surface. Invariance of Euler characteristic and orientability under homeomorphism. The standard list gives classification of surfaces up to a homeomorphism.

## VI. GEOMETRY, CURVATURE AND SYMMETRY

24. Internal geometry on a surface. Riemannian metric; isometry. Why angles are easier to measure than distances. Metric and Riemannian metric induced by embedding. Their difference. How a Riemannian metric allows to integrate functions on a surface (the volume element). Gradient vector field for a function and connection between third and fourth guises for the Euler characteristic.

25. Examples of surfaces with Riemannian metrics. The round sphere; the ellipsoids; cones and cylinders; torus of revolution (the “bagel”). Flat torus, the first example of Riemannian metric which does not arise from embedding into three-dimensional space (but it arises from an embedding to four-dimensional space).

26. Geodesics: existence and basic properties including local uniqueness between two points. Various ways global uniqueness fails.

27. Gaussian curvature. Gauss-Bonnet theorem: the fifth guise of Euler characteristic– the integral of Gaussian curvature divided by  $2\pi$ .

28. Metrics of constant curvature on surfaces: positive – sphere and projective plane; zero – torus and Klein bottle; negative? Construction of a metric of constant negative curvature on the plane (Poincare upper half plane). Existence of transitive group of isometries.

29. Construction of metrics of constant negative curvature on surfaces with negative Euler characteristic. Examples of Fuchsian groups.

## VII. FUNCTIONS OF COMPLEX VARIABLE AND RIEMANN SURFACES

30. One dimensional complex manifolds as a particular case of surfaces. Their orientability. Simple examples: Riemann sphere and flat torus. More complicated examples: surfaces constructed in the previous section.

31. Difficulties with definition of domains for elementary functions of a complex variable such as a root or a logarithm. Construction of a Riemann surface for holomorphic function. Examples: algebraic functions, roots from polynomials and suchlike.

32. The sixth algebraic guise of Euler characteristic. Construction of modular surface.

## VIII. BACK TO SURFACES IN EUCLIDEAN SPACE

33. Principle of a local calculation up to an appropriate order. The first and second quadratic forms of an embedded surface. Intrinsic character of the first form.

34. Curvature of an embedded surface in a direction. Principal curvatures. Gaussian curvature equals to the product of principal curvatures.

35. Pseudosphere, an embedded (piece of) surface of constant negative curvature. Discussion of embeddings for surfaces of constant positive, zero and negative curvature.

## TEXTS

1. Chapters from the forthcoming book: A.Katok and A. Sossinsky, *Introduction to modern geometry and topology*; will be made available to students.
2. Chapters from H.M.S. Coxeter, *Introduction to Geometry*, John Wiley, 1989 or earlier editions.
3. Chapters from *Flavors of geometry*, Silvio Levy, ed., Cambridge University Press, 1997.
4. Chapters from S.Katok, *Fuchsian Groups*, The University of Chicago Press, 1992.