MASS-09; ALGEBRA

FALL 2009

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HOMEWORK # 3

Due on Wednesday, September 23

13. Consider the following group Q with 8 elements: ± 1, $\pm i, \pm j, \pm k$ with the multiplication rules: $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$; multiplication by -1 follows usual rules, in particular -1 commutes with all elements.

(1) Embed Q into the group $Isom^+\mathbb{R}^4$ of orientation preserving isometries of $\mathbb{R}^4$.
(2) Prove that Q cannot be embedded into $Isom^+\mathbb{R}^3$.

14. Show that there are exactly five different groups of order 8 : $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, D_4, and Q.

15. The factor–group $\mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{Z}^2$ is the lattice of all vectors with integer coordinates, is called the *torus* and is denoted by $\mathbb{T}^2$. The (Euclidean) distance on the torus is defined as the minimum of Euclidean distance between the elements of corresponding cosets.

Describe the group of isometries of the torus with Euclidean distance.

16. *Elliptic plane* $\mathbb{E}^2$ is obtained from the sphere S^2 by identifying pairs of opposite points. The distance between the pairs $(x, -x)$ and $(y, -y)$ is defined as $\min(d(x, y), d(x, -y))$, where d is distance on the sphere (the angular distance). Lines on the elliptic plane are defined as images of great circles on the sphere under identifications. Thus any two lines on the elliptic plane intersect at exactly one point.

(1) Prove that the group of isometries of elliptic plane, $Isom(\mathbb{E}^2)$, is isomorphic to the group $SO(3, \mathbb{R})$ of direct isometries of the sphere.
(2) Prove that any isometry of $\mathbb{E}^2$ has a fixed point.