# MASS-09; ALGEBRA

FALL 2009

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HOMEWORK # 4

Due on Wednesday September 30

**17.** Find a necessary and sufficient condition for the product of two screw motions to be a translation.

**18.** Let $G$ be the isometry group of the $n$-dimensional cube $I^n$.

(1) Find the order of $G$.

(2) Find the numbers of even and odd involutions (elements of order two) in $G$.

*Hint: Use induction in dimension.*

**19.** The *octacube* whose imaginative rendering in stainless steel you may see in the lobby next to your classroom, is the four-dimensional *polytope* with 24 vertices: $(\pm 1, \pm 1, \pm 1, \pm 1)$, $(\pm 2, 0, 0, 0)$, $(0, \pm 2, 0, 0)$, $(0, 0, \pm 2, 0)$, $(0, 0, 0, \pm 2)$. Vertices of the octacube are obtained by adding to the vertices of the cube images of its center under reflections in its three-dimensional faces.

(1) Prove that the isometry group of the octacube acts transitively on its vertices and three-dimensional faces and that thus octacube qualifies as a *regular polytope*.
*Hints: (i) Construct an isometry that maps* $(2, 0, 0, 0)$ *to* $(1, 1, 1, 1)$*;*
*(ii) It may help to construct the dual polytope.*

(2) Carry the similar construction in $\mathbb{R}^3$, draw resulting polyhedron, find its isometry group and explain why it is not regular.

**20.** Prove that for any $n$ there exists a discrete infinite subgroup of $Isom(\mathbb{R}^3)$ that contains an element of order $n$.

**21.** Prove that any finite group of affine transformations in $\mathbb{R}^3$ is isomorphic to a group of isometries.