7

Isometry and similarity in Euclidean space

This chapter is the three-dimensional counterpart of Chapters 3 and 5. We have had to discuss circles and spheres first in order to be able to make use of the sphere of Apollonius (6.81), which provides the only known construction for the invariant point of an arbitrary similarity. In § 7.5 we find a simple proof for the well-known kinematical theorem that every motion is a screw displacement. In § 7.6 we see that every similarity (except the screw displacement and glide reflection, which have no invariant point) may be regarded as a special case of a three-dimensional *spiral* similarity.

Most isometries are familiar in everyday life. When you walk straight forward you are undergoing a translation. When you turn a corner, it is a rotation; when you ascend a spiral staircase, a screw displacement. The transformation that interchanges yourself and your image in an ordinary mirror is a reflection, and it is easy to see how you could combine this with a rotation or a translation to obtain a rotatory reflection or a glide reflection, respectively.

7.1 DIRECT AND OPPOSITE ISOMETRIES

> *A congruence is either proper, carrying a left screw into a left and a right one into a right, or it is improper or reflexive, changing a left screw into a right one and vice versa. The proper congruences are those transformations which . . . connect the positions of points of a rigid body before and after a motion.*
>
> H. Weyl [**1**, pp. 43-44]

The axioms of congruence, a sample of which was given in 1.26, can be extended in a natural manner from plane geometry to solid geometry. In space, an *isometry* (Weyl's "congruence") is still any transformation that preserves length, so that a line segment PQ is transformed into a congruent segment $P'Q'$. The most familiar examples are the *rotation* about a given line through a given angle and the *translation* in a given direction through a given distance. In the former case the axis of rotation has all its points invariant; in the latter there is no invariant point, except when the distance is zero so that the translation is the identity. A *reflection* is the special kind of isometry which has a whole plane of invariant points: the mirror. By a simple argument involving three spheres instead of two circles, we can easily prove the following analogue of Theorem 2.31:

7.11 *If an isometry has three non-collinear invariant points, it must be either the identity or a reflection.*

When two tetrahedra $ABCP$, $ABCP'$ are images of each other by reflection in their common face, we may regard the "broken line" formed by the three edges AB, BC, CP as a kind of rudimentary screw, and the image formed by AB, BC, CP' as an oppositely oriented screw: if one is right-handed the other is left-handed. A model is easily made from two pieces of stiff wire, with right-angled bends at B and C. In this manner the idea of *sense* can be extended from two dimensions to three: we can say whether two given congruent tetrahedra agree or disagree in sense. In the former case we shall find that either tetrahedron can be *moved* (like a screw in its nut) to the position previously occupied by the other.

This distinction arises in analytic geometry when we make a coordinate transformation. If O is the origin and X, Y, Z are at unit distances along the positive coordinate axes, the sense of the tetrahedron $OXYZ$ determines whether the system of axes is right-handed or left-handed. (A coordinate transformation determines an isometry transforming each point (x, y, z) into the point that has the *same* coordinates in the new system.)

Since an isometry is determined by its effect on a tetrahedron,

7.12 *Any two congruent tetrahedra $ABCD$, $A'B'C'D'$ are related by a unique isometry $ABCD \rightarrow A'B'C'D'$, which is direct or opposite according as the sense of $A'B'C'D'$ agrees or disagrees with that of $ABCD$.*

(Some authors, such as Weyl, say "proper or improper" instead of "direct or opposite.")

The solid analogue of Theorem 3.12 is easily seen to be:

7.13 *Two given congruent triangles are related by just two isometries: one direct and one opposite.*

As a counterpart for 3.13 we have [Coxeter **1**, p. 36]:

7.14 *Every isometry is the product of at most four reflections. If there is an invariant point, "four" can be replaced by "three."*

Since a reflection reverses sense, an isometry is direct or opposite according as it is the product of an even or odd number of reflections: 2 or 4 in the former case, 1 or 3 in the latter. In particular, a direct isometry with an invariant point is the product of just two reflections, and since the two mirrors have a common point they have a common line. Hence

7.15 *Every direct isometry with an invariant point is a rotation.*

Also, as Euler observed in 1776,

7.16 *The product of two rotations about lines through a point O is another such rotation.*

EXERCISE

The product of rotations through π about two intersecting lines that form an angle α is a rotation through 2α.

7.2 THE CENTRAL INVERSION

One of the most important opposite isometries is the *central inversion* (or "reflection in a point"), which transforms each point P into the point P' for which the midpoint of PP' is a fixed point O. This can be described as the product of reflections in any three mutually perpendicular planes through O. Taking these three mirrors to be the coordinate planes $x = 0, y = 0, z = 0$, we see that the central inversion in the origin transforms each point (x, y, z) into $(-x, -y, -z)$.

The name "central inversion," though well established in the literature of crystallography, is perhaps unfortunate: we must be careful to distinguish it from inversion in a sphere.

For most purposes the central inversion plays the same role in three dimensions as the half-turn in two. But we must remember that, since 3 is an odd number, the central inversion is an opposite isometry whereas the half-turn is direct. In space, the name *half-turn* is naturally used for the rotation through π about a line (or the "reflection in a line"), which is still direct [Lamb **1**, p. 9].

EXERCISE

What is the product of half-turns about three mutually perpendicular lines through a point?

7.3 ROTATION AND TRANSLATION

The treatment of translation in § 3.2 can be adapted to three dimensions by defining a translation as the product of two central inversions. We soon see that either the first center or the second may be arbitrarily assigned, and that the two inversions may be replaced by two half-turns about parallel axes or by two reflections in parallel mirrors.

Thus the product of two reflections is either a translation or a rotation.

The latter arises when the two mirrors intersect in a line, the axis of the rotation. In particular, the product of reflections in two perpendicular mirrors is a half-turn.

The product of reflections in two planes through a line l, being a rotation about l, is the same as the product of reflections in two other planes through l making the same dihedral angle as the given planes (in the same sense). Similarly, the product of reflections in two parallel planes, being a translation, is the same as the product of reflections in two other planes parallel to the given planes and having the same distance apart.

EXERCISE

What is the product of reflections in three parallel planes?

7.4 THE PRODUCT OF THREE REFLECTIONS

The three simplest kinds of isometry, namely rotation, translation and reflection, combine in commutative pairs to form the *screw displacement, glide reflection* and *rotatory reflection.* A screw displacement is the product of a rotation with a translation along the direction of the axis. A glide reflection is the product of a reflection with a translation along the direction of a line lying in the mirror, that is, the product of reflections in three planes of which two are parallel while the third is perpendicular to both. A rotatory reflection is the product of a reflection with a rotation whose axis is perpendicular to the mirror. When this rotation is a half-turn, the rotatory reflection reduces to a central inversion.

Any rotatory reflection can be analysed into a central inversion and a residual rotation. For, if the rotation involved in the rotatory reflection is a rotation through θ, we may regard it as the product of a half-turn and a rotation through $\theta + \pi$ (or $\theta - \pi$). Thus a rotatory reflection can just as well be called a *rotatory inversion:* the product of a central inversion and a rotation whose axis passes through the center.

Any opposite isometry T that has an invariant point O is either a single reflection or the product of reflections in three planes through O. Its product TI with the central inversion in O, being a direct isometry with an invariant point, is simply a rotation S about a line through O. Hence the given opposite isometry is the rotatory inversion

$$\mathrm{T} = \mathrm{SI}^{-1} = \mathrm{SI}:$$

7.41 *Every opposite isometry with an invariant point is a rotatory inversion.*

Since three planes that have no common point are all perpendicular to one plane α, the reflections in them (as applied to a point in α) behave like the reflections in the lines that are their sections by α. Thus we can make use of Theorem 3.31 and conclude that

7.42 *Every opposite isometry with no invariant point is a glide reflection.*

EXERCISES

1. What is the product of reflections in three planes through a line?

2. Let ABC and $A'B'C'$ be two congruent triangles in distinct planes. Consider the perpendicular bisectors of AA', BB', CC'. If these three planes have just one common point O, the two triangles are related by a rotatory inversion with center O. (*Hint:* If they were related by a rotation, the three planes would intersect in a line.)

3. Every opposite isometry is expressible as the product of a reflection and a half-turn.

7.5 SCREW DISPLACEMENT

The only remaining possibility is a direct isometry with no invariant point. Let S be any direct isometry (with or without an invariant point), transforming an arbitrary point A into A'. Let R_1 be the reflection that interchanges A and A'. Then the product R_1S is an opposite isometry leaving A' invariant. By 7.41, this is a rotatory inversion or rotatory reflection $R_2R_3R_4$, the product of a rotation R_2R_3 and a reflection R_4, the mirror for R_4 being perpendicular to the axis for R_2R_3. Since this rotation may be expressed as the product of two reflections in various ways (§7.3), we can adjust the mirrors for R_2 and R_3 so as to make the former perpendicular to the mirror for R_1. Since both these planes remain perpendicular to the mirror for R_4, we now have

$$S = R_1R_2R_3R_4,$$

the product of the two rotations R_1R_2, R_3R_4, both of which are half-turns [Veblen and Young **2**, p. 318]:

7.51 *Every direct isometry is expressible as the product of two half-turns.*

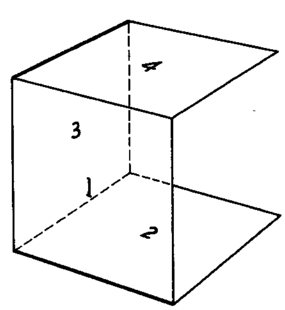

Figure 7.5a

If the isometry has an invariant point, it is a rotation, which may be expressed in various ways as the product of half-turns about two intersecting lines. When there is no invariant point, the axes of the two half-turns are either parallel, in which case the product is a translation, or *skew,* like two opposite edges of a tetrahedron. Two skew lines always lie in a pair of parallel planes, namely, the plane through each line parallel to the other.

Since a half-turn is the product of reflections in *any* two perpendicular lines through its axis, the two half-turns R_1R_2, R_3R_4 with skew axes are respectively equal to $R'_1R'_2$, $R'_3R'_4$, where the mirrors for R'_2 and R'_4 are parallel while the other two are perpendicular to them (Figure 7.5*a*). Hence

$$R_1R_2R_3R_4 = R'_1R'_2R'_3R'_4 = R'_1R'_3R'_2R'_4,$$

where the interchange of the middle reflections is possible since the half-turn $R'_2R'_3$ may be equally well expressed as $R'_3R'_2$. We have now fulfilled our purpose of expressing the general direct isometry as a screw displacement: the product of the rotation $R'_1R'_3$ and the translation $R'_2R'_4$ along the axis of the rotation. (This axis meets both the skew lines at right angles, and therefore measures the shortest distance between them.) In other words,

7.52 *Every displacement is either a rotation or a translation or a screw displacement.*

(For an alternative treatment see Thomson and Tait [**1**, § 102].)

EXERCISES

1. What kind of isometry transforms the point (x, y, z) into
(a) $(x, y, -z)$, (b) $(-y, x, z)$, (c) $(x, y, z + 1)$,
(d) $(-y, x, z + 1)$, (e) $(-x, y, z + 1)$, (f) $(-y, x, -z)$?

2. The product of half-turns about two skew lines at right angles is a screw displacement, namely, the product of a half-turn about the line of shortest distance and a translation through twice this shortest distance [Lamb **1**, p. 11, Ex. 6].

7.6 SPIRAL SIMILARITY

It can be proved by elementary methods that every Euclidean similarity other than a rigid motion has a fixed point.

Hilbert and Cohn-Vossen [**1**, p. 331]

In Euclidean space, the definition of *dilatation* is exactly the same as in the plane. In fact, § 5.1 can be applied, word for word, to three dimensions, except that the special dilatation $AB \to BA$ or $O(-1)$ is not a half-turn but a central inversion (§ 7.2). Likewise, § 5.2 applies to spheres just as well as to circles: Figure 5.2*a* may be regarded as a plane section of two unequal spheres with their centers C, C' and their centers of similitude O, O_1. Two equal spheres are related by a translation and by a central inversion.

However, an important difference appears when we consider questions of sense. In the plane, every dilatation is direct, but in space the dilatation $O(\mu)$ is direct or opposite according as μ is positive or negative; for example,