we say that S is of *period* 4. Similarly S^2, being a half-turn, is of period 2 [see Coxeter **1**, p. 39]. The only transformation of period 1 is the identity. A translation is aperiodic (that is, it has no period), but it is conveniently said to be of infinite period.

Some figures admit both reflections and rotations as symmetry operations. The letter H (Figure 2.4*d*) has a horizontal mirror (like E) and a vertical mirror (like A), as well as a center of rotational symmetry (like N) where the two mirrors intersect. Thus it has four symmetry operations: the identity 1, the horizontal reflection R_1, the vertical reflection R_2, and the half-turn $R_1R_2 = R_2R_1$.

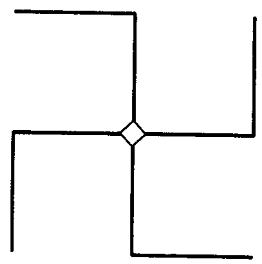

Figure 2.4c

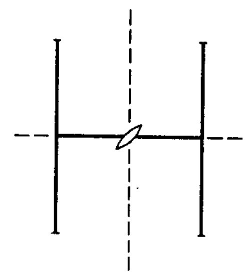

Figure 2.4d

EXERCISES

1. Every isometry of period 2 is either a reflection or a half-turn [Bachmann **1**, pp. 2–3].

2. Express (*a*) a half-turn, (*b*) a quarter-turn, as transformations of (i) Cartesian coordinates, (ii) polar coordinates. (Take the origin to be the center of rotation.)

2.5 GROUPS

Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection.

Hermann Weyl [**1**, p. 5]

A set of transformations [Birkhoff and MacLane **1**, pp. 119–122] is said to form a *group* if it contains the inverse of each and the product of any two (including the product of one with itself or with its inverse). The number of distinct transformations is called the *order* of the group. (This may be either finite or infinite.) Clearly the symmetry operations of any figure form a group. This is called the *symmetry group* of the figure. In the extreme case when the figure is completely irregular (like the numeral 6) its symmetry group is of order one, consisting of the identity alone.

The symmetry group of the letter E or A (Figure 2.4*a*) is the so-called *dihedral* group of order 2, generated by a single reflection and denoted by D_1. (The name is easily remembered, as the Greek origin of the word "dihedral" is almost equivalent to the Latin origin of "bilateral.") The symmetry group of the letter N (Figure 2.4*b*) is likewise of order 2, but in this case the generator is a half-turn and we speak of the *cyclic* group, C_2. The two groups D_1 and C_2 are abstractly identical or *isomorphic;* they are different geometrical representations of the single abstract group of order 2, defined by the relation

2.51 $$R^2 = 1$$

or $R = R^{-1}$ [Coxeter and Moser **1**, p. 1].

The symmetry group of the swastika is C_4, the cyclic group of order 4, generated by the quarter-turn S and abstractly defined by the relation $S^4 = 1$. That of the letter H (Figure 2.4*d*) is D_2, the dihedral group of order 4, generated by the two reflections R_1, R_2 and abstractly defined by the relations

2.52 $$R_1{}^2 = 1, \quad R_2{}^2 = 1, \quad R_1R_2 = R_2R_1.$$

Although C_4 and D_2 both have order 4, they are *not* isomorphic: they have a different structure, different "multiplication tables." To see this, it suffices to observe that C_4 contains two operations of period 4, whereas all the operations in D_2 (except the identity) are of period 2: the generators obviously, and their product also, since

$$(R_1R_2)^2 = R_1R_2R_1R_2 = R_1R_2R_2R_1 = R_1R_2{}^2R_1 = R_1R_1 = R_1{}^2 = 1.$$

This last remark illustrates what we mean by saying that 2.52 is an *abstract definition* for D_2, namely that every true relation concerning the generators R_1, R_2 is an algebraic consequence of these simple relations. An alternative abstract definition for the same group is

2.53 $$R_1{}^2 = 1, \quad R_2{}^2 = 1, \quad (R_1R_2)^2 = 1,$$

from which we can easily deduce $R_1R_2 = R_2R_1$.

The general cyclic group C_n, of order n, has the abstract definition

2.54 $$S^n = 1.$$

Its single generator S, of period n, is conveniently represented by a rotation through $360°/n$. Then S^k is a rotation through k times this angle, and the n operations in C_n are given by the values of k from 1 to n, or from 0 to $n - 1$. In particular, C_5 occurs in nature as the symmetry group of the periwinkle flower.

EXERCISE

Express a rotation through angle α about the origin as a transformation of (i) polar coordinates, (ii) Cartesian coordinates. If $f(r, \theta) = 0$ is the equation for a curve in polar coordinates, what is the equation for the transformed curve?

2.6 THE PRODUCT OF TWO REFLECTIONS

Thou in thy lake dost see
Thyself.

J. M. Legaré (1823-1859)
(To a Lily)

In any group of transformations, the associative law

$$(RS)T = R(ST)$$

is automatically satisfied, but the commutative law

$$RS = SR$$

does not necessarily hold, and care must be taken in inverting a product, for example,

$$(RS)^{-1} = S^{-1}R^{-1},$$

not $R^{-1}S^{-1}$. (This becomes clear when we think of R and S as the operations of putting on our socks and shoes, respectively.)

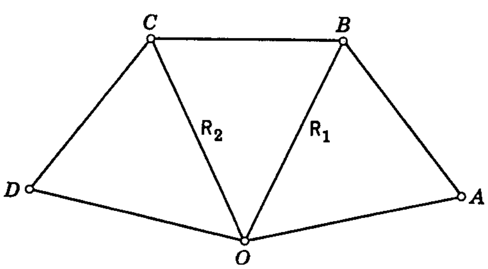

Figure 2.6a

The product of reflections in two intersecting lines (or planes) is a rotation through twice the angle between them. In fact, if $A, B, C, D, \ldots$ are evenly spaced on a circle with center O, let R_1 and R_2 be the reflections in OB and OC (Figure 2.6*a*). Then R_1 reflects the triangle OAB into OCB, which is reflected by R_2 to OCD; thus R_1R_2 is the rotation through $\angle AOC$ or $\angle BOD$, which is twice $\angle BOC$. Since a rotation is completely determined by its center and its angle, R_1R_2 is equal to the product of reflections in any two lines through O making the same angle as OB and OC. (The reflections in OA and OB are actually $R_1R_2R_1$ and R_1, whose product is $R_1R_2R_1{}^2 = R_1R_2$.) In particular, the half-turn about O is the product of reflections in any two perpendicular lines through O.

Since R_1R_2 is a counterclockwise rotation, R_2R_1 is the corresponding clockwise rotation; in fact,

$$R_2R_1 = R_2{}^{-1}R_1{}^{-1} = (R_1R_2)^{-1}.$$

This is the same as R_1R_2 if the two mirrors are at right angles, in which case R_1R_2 is a half-turn and $(R_1R_2)^2 = 1$.

EXERCISES

1. The product of quarter-turns (in the same sense) about C and B is the half-turn about the center of a square having BC for a side.

2. Let $ACPQ$ and $BARS$ be squares on the sides AC and BA of a triangle ABC. If B and C remain fixed while A varies freely, PS passes through a fixed point.

2.7 THE KALEIDOSCOPE

D_2 is a special case of the general dihedral group D_n, which is, for $n > 2$, the symmetry group of the regular n-gon, $\{n\}$. (See Figure 2.7*a* for the cases $n = 3, 4, 5$.) This is evidently a group of order $2n$, consisting of n rotations (through the n effectively distinct multiples of $360°/n$) and n reflections. When n is odd, each of the n mirrors joins a vertex to the midpoint of the opposite side; when n is even, $\frac{1}{2}n$ mirrors join pairs of opposite vertices and $\frac{1}{2}n$ bisect pairs of opposite sides [see Birkhoff and MacLane **1**, pp. 117–118, 135].

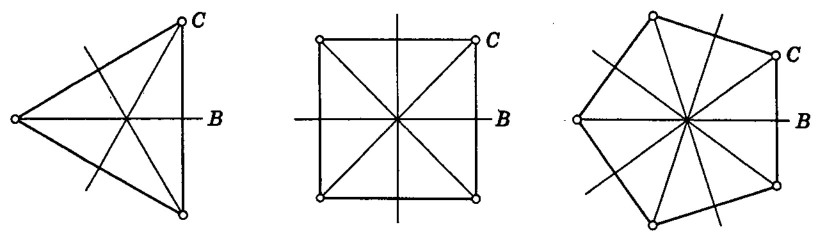

Figure 2.7a

The n rotations are just the operations of the cyclic group C_n. Thus the operations of D_n include all the operations of C_n: in technical language, C_n is a *subgroup* of D_n. The rotation through $360°/n$, which generates the subgroup, may be described as the product $S = R_1R_2$ of reflections in two adjacent mirrors (such as OB and OC in Figure 2.7*a*) which are inclined at $180°/n$.

Let $R_1, R_2, \ldots, R_n$ denote the n reflections in their natural order of arrangement. Then R_1R_{k+1}, being the product of reflections in two mirrors inclined at k times $180°/n$, is a rotation through k times $360°/n$:

$$R_1R_{k+1} = S^k.$$

Thus $R_{k+1} = R_1S^k$, and the n reflections may be expressed as

$$R_1, R_1S, R_1S^2, \ldots, R_1S^{n-1}.$$

In other words, D_n is generated by R_1 and S. By substituting R_1R_2 for S, we see that the same group is equally well generated by R_1 and R_2, which satisfy the relations

2.71 $$R_1^2 = 1, \quad R_2^2 = 1, \quad (R_1R_2)^n = 1.$$

(The first two relations come from 2.51 and the third from 2.54.) These relations can be shown to suffice for an abstract definition [see Coxeter and Moser **1**, pp. 6, 36].

A practical way to make a model of D_n is to join two ordinary mirrors by a hinge and stand them on the lines *OB, OC* of Figure 2.7*a* so that they are inclined at $180°/n$. Any object placed between the mirrors yields $2n$ visible images (including the object itself). If the object is your right hand, n of the images will look like a left hand, illustrating the principle that, since a reflection reverses sense, the product of any even number of reflections preserves sense, and the product of any odd number of reflections reverses sense.

The first published account of this instrument seems to have been by Kircher in 1646. The name *kaleidoscope* (from καλος, beautiful; ειδος, a form; and σκοπειν, to see) was coined by Sir David Brewster, who wrote a treatise on its theory and history. He complained [Brewster **1**, p. 147] that Kircher allowed the angle between the two mirrors to be any submultiple of 360° instead of restricting it to submultiples of 180°.

The case when $n = 2$ is, of course, familiar. Standing between two perpendicular mirrors (as at a corner of a room), you see your image in each and also the image of the image, which is the way other people see you.

Having decided to use the symbol D_n for the dihedral group generated by reflections in two planes making a "dihedral" angle of $180°/n$, we naturally stretch the notation so as to allow the extreme value $n = 1$. Thus D_1 is the group of order 2 generated by a single reflection, that is, the symmetry group of the letter E or A, whereas the isomorphic group C_2, generated by a half-turn, is the symmetry group of the letter N.

According to Weyl [**1**, pp. 66, 99], it was Leonardo da Vinci who discovered that the only finite groups of isometries in the plane are

$$C_1, C_2, C_3, \ldots,$$
$$D_1, D_2, D_3, \ldots.$$

His interest in them was from the standpoint of architectural plans. Of course, the prevalent groups in architecture have always been D_1 and D_2. But the pyramids of Egypt exhibit the group D_4, and Leonardo's suggestion has been followed to some extent in modern times: the Pentagon Building in Washington has the symmetry group D_5, and the Bahai Temple near Chicago has D_9. In nature, many flowers have dihedral symmetry groups such as D_6. The symmetry group of a snowflake is usually D_6 but occasionally only D_3. [Kepler **1**, pp. 259–280.]

If you cut an apple the way most people cut an orange, the core is seen to have the symmetry group D_5. Extending the five-pointed star by straight cuts in each half, you divide the whole apple into ten pieces from each of which the core can be removed in the form of two thin flakes.

EXERCISES

1. Describe the symmetry groups of

(a) a scalene triangle, (b) an isosceles triangle,

(c) a parabola, (d) a parallelogram,
(e) a rhombus, (f) a rectangle,
(g) an ellipse.

2. Use inverses and the associative law to prove algebraically the "cancellation rule" which says that the relation

$$RT = ST$$

implies R = S.

3. Show how the usual defining relations for D_3, namely 2.71 with $n = 3$, may be deduced by algebraic manipulation from the simpler relations

$$R_1^2 = 1, \quad R_1R_2R_1 = R_2R_1R_2.$$

4. The cyclic group C_m is a subgroup of C_n if and only if the number m is a divisor of n. In particular, if n is prime, the only subgroups of C_n are C_n itself and C_1.

2.8 STAR POLYGONS

Instead of deriving the dihedral group D_n from the regular polygon $\{n\}$, we could have derived the polygon from the group: the vertices of the polygon are just the n images of a point P_0 (the C of Figure 2.7*a*) on one of the two mirrors of the kaleidoscope. In fact, there is no need to use the whole group D_n: its subgroup C_n will suffice. The vertex P_k of the polygon $P_0P_1 \ldots P_{n-1}$ can be derived from the initial vertex P_0 by a rotation through k times $360°/n$.

More generally, rotations about a fixed point O through angles θ, 2θ, 3θ, ... transform any point P_0 (distinct from O) into other points P_1, P_2, P_3, ... on the circle with center O and radius OP_0. In general, these points become increasingly dense on the circle; but if the angle θ is commensurable with a right angle, only a finite number of them will be distinct. In particular, if $\theta = 360°/n$, where n is a positive integer greater than 2, then there will be n points P_k whose successive joins

$$P_0P_1, P_1P_2, \ldots, P_{n-1}P_0$$

are the sides of an ordinary regular n-gon.

Let us now extend this notion by allowing n to be any rational number greater than 2, say the fraction p/d (where p and d are coprime). Accordingly, we define a (generalized) *regular polygon* $\{n\}$, where $n = p/d$. Its p vertices are derived from P_0 by repeated rotations through $360°/n$, and its p sides (enclosing the center d times) are

$$P_0P_1, P_1P_2, \ldots, P_{p-1}P_0.$$

Since a ray coming out from the center without passing through a vertex will cross d of the p sides, this denominator d is called the *density* of the polygon [Coxeter **1**, pp. 93–94]. When $d = 1$, so that $n = p$, we have the ordinary regular p-gon, $\{p\}$. When $d > 1$, the sides cross one another, but the crossing points are not counted as vertices. Since d may be any positive integer relatively prime to p and less than $\frac{1}{2}p$, there is a regular polygon $\{n\}$ for each rational number $n > 2$. In fact, it is occasionally desirable to include also the *digon* $\{2\}$, although its two sides coincide.

When $p = 5$, we have the pentagon $\{5\}$ of density 1 and the *pentagram* $\{\frac{5}{2}\}$ of density 2, which was the Pythagorean symbol of good health [Ball **1**, p. 248]. Similarly, the *octagram* $\{\frac{8}{3}\}$ and the *decagram* $\{\frac{10}{3}\}$ have density 3, while the *dodecagram* $\{\frac{12}{5}\}$ has density 5 (Figure 2.8*a*). These particular polygons have names as well as symbols because they occur as faces of interesting polyhedra and tessellations.*

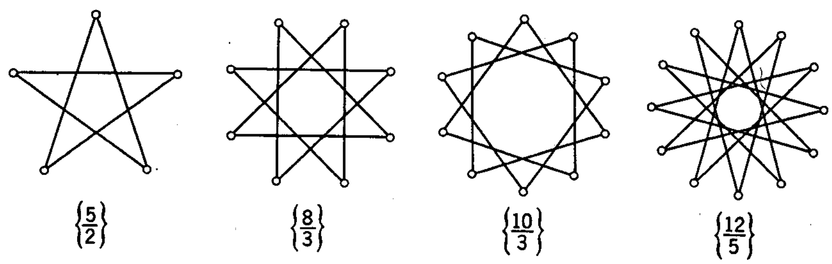

Figure 2.8a

Polygons for which $d > 1$ are known as *star* polygons. They are frequently used in decoration. The earliest mathematical discussion of them was by Thomas Bradwardine (1290–1349), who became archbishop of Canterbury for the last month of his life. They were also studied by the great German scientist Kepler (1571–1630) [see Coxeter **1**, p. 114]. It was the Swiss mathematician L. Schläfli (1814–1895) who first used a numerical symbol such as $\{p/d\}$. This notation is justified by the occurrence of formulas that hold for $\{n\}$ equally well whether n be an integer or a fraction. For instance, any side of $\{n\}$ forms with the center O an isosceles triangle OP_0P_1 (Figure 2.8*b*) whose angle at O is $2\pi/n$. (As we are introducing trigonometrical ideas, it is natural to use radian measure and write 2π instead of 360°.) The base of this isosceles triangle, being a side of the polygon, is conveniently denoted by $2l$. The other sides of the triangle are equal to the circumradius R of the polygon. The altitude or median from O is the inradius r of the polygon. Hence

2.81 $$R = l \csc \frac{\pi}{n}, \qquad r = l \cot \frac{\pi}{n}.$$

If $n = p/d$, the area of the polygon is naturally defined to be the sum of the areas of the p isosceles triangles, namely

* H. S. M. Coxeter, M. S. Longuet-Higgins, and J. C. P. Miller, Uniform polyhedra, *Philosophical Transactions of the Royal Society,* **A, 246** (1954), pp. 401–450.

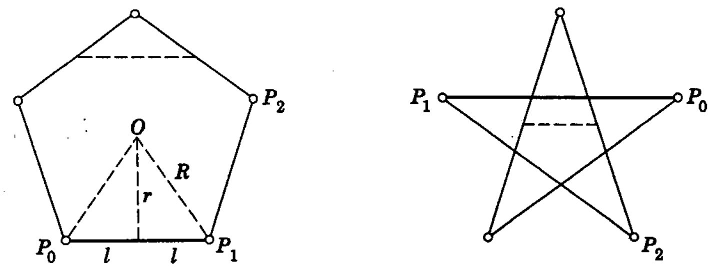

Figure 2.8b

2.82 $$plr = pl^2 \cot \frac{\pi}{n}.$$

When $d = 1$, this is simply $pl^2 \cot \pi/p$; in other cases our definition of area has the effect that every part of the interior is counted a number of times equal to the "local density" of that part; for example, the pentagonal region in the middle of the pentagram $\{\frac{5}{2}\}$ is counted twice.

The angle $P_0P_1P_2$ between two adjacent sides of $\{n\}$, being the sum of the base angles of the isosceles triangle, is the supplement of $2\pi/n$, namely

2.83 $$\left(1 - \frac{2}{n}\right)\pi.$$

The line segment joining the midpoints of two adjacent sides is called the *vertex figure* of $\{n\}$. Its length is clearly

2.84 $$2l \cos \frac{\pi}{n}$$

[Coxeter **1**, pp. 16, 94].

EXERCISES

1. If the sides of a polygon inscribed in a circle are all equal, the polygon is regular.

2. If a polygon inscribed in a circle has an odd number of vertices, and all its angles are equal, the polygon is regular. (Marcel Riesz.)

3. Find the angles of the polygons

$$\{5\}, \quad \{\tfrac{5}{2}\}, \quad \{9\}, \quad \{\tfrac{9}{2}\}, \quad \{\tfrac{9}{4}\}.$$

4. Find the radii and vertex figures of the polygons

$$\{8\}, \quad \{\tfrac{8}{3}\}, \quad \{12\}, \quad \{\tfrac{12}{5}\}.$$

5. Give polar coordinates for the kth vertex P_k of a polygon $\{n\}$ of circumradius 1 with its center at the pole, taking P_0 to be $(1, 0)$.

6. Can a square cake be cut into nine slices so that everyone gets the same amount of cake and the same amount of icing?