$\angle MAL$ is positive). Hence there must be some intermediate position for which

$$\angle DAL = \angle ADB.$$

(To be precise, we can apply Dedekind's axiom 12.51 to the points on BM satisfying the two opposite inequalities.) For such a point D (Figure 15.2*f*) we obtain two triangles OAE, ODF by drawing EF perpendicular to BD through O, the midpoint of AD. Since these triangles are congruent, EF is perpendicular not only to BD but also to AL.

Nonintersecting lines that are not parallel are said to be *ultraparallel* (or "hyperparallel"). We are not asserting the existence of such lines, but merely showing how they must behave if they do exist.

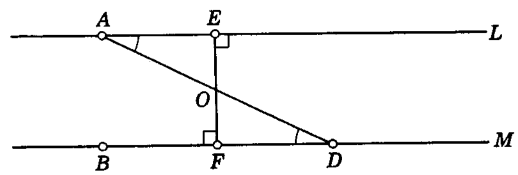

Figure 15.2f

EXERCISES

1. Prove 15.25 without referring to Carslaw **1**.
2. Give a complete proof that, if two lines have a common perpendicular, they do not intersect.
3. Example 4 on p. 16 remains valid when A is an end so that the triangle is asymptotic.

15.3 ISOMETRY

Beside the actual universe I can set in imagination other universes in which the laws are different.

J. L. Synge [**2**, p. 21]

The whole theory of finite groups of isometries (§§ 2.3–3.1) belongs to absolute geometry, because it is concerned with isometries having at least one invariant point. The first departure from our previous treatment (§ 3.2) is in the discussion of isometries without invariant points. We must now distinguish between a *translation,* which is the product of half-turns about two distinct points, and a *parallel displacement,* which is the product of reflections in two parallel lines.

The product of half-turns about two distinct points O, O' is a translation along a given line (called the *axis* of the translation) in a given sense through a given distance, namely, along OO' in the sense of the ray O'/O through the distance $2OO'$. Since a translation is determined by its axis and directed distance, the product of half-turns about O, O' is the same as the product of half-turns about Q, Q', provided the directed segment QQ' is congruent to OO' *on the same line* (Figure 3.2*a*). If P is on this line, the distance PP^T is just twice OO'. (If not, it may be greater!)

By the argument used in proving 3.21, the product of two translations with the same axis, or with intersecting axes, is a translation. (It is only in the former case that we can be sure of commutativity.) More precisely, we have

15.31 (Donkin's theorem*) *The product of three translations along the directed sides of a triangle, through twice the lengths of these sides, is the identity.*

We shall see later that the product of two translations with nonintersecting axes may be a rotation.

By the argument used in proving 3.22, if two lines have a common perpendicular, the product of reflections in them is a translation along this common perpendicular through twice the distance between them. (Such lines may be either parallel or ultraparallel according to the nature of the geometry.)

Again, as in 3.13, every isometry is the product of at most three reflections. If the isometry is direct, the number of reflections is even, namely 2. It follows from 15.26 that

15.32 *Every direct isometry* (*of the plane*) *with no invariant point is either a parallel displacement or a translation.*

It is remarkable that absolute geometry includes the whole theory of glide reflection. The only changes needed in the previous treatment (§ 3.3) are where the word "parallel" was used. (In Figure 3.3*b* we must define m, m' as being perpendicular to OO'; they are not necessarily parallel to each other.) As an immediate application of these ideas we have Hjelmslev's theorem, which is one of the best instances of a genuinely surprising result belonging to absolute geometry. The treatment in § 3.6 remains valid without changing a single word!

Likewise, the one-dimensional groups of § 3.7 belong to absolute geometry, the only change being that again the mirrors m, m' (Figure 3.7*b*) should not be said to be "parallel" but both perpendicular to the same (horizontal) line. On the other hand, the whole theory of lattices (Chapter 4) and of similarity (Chapter 5) must be abandoned.

The extension of absolute geometry from two dimensions to three presents no difficulty. In particular, much of the Euclidean theory of isometry (§ 7.1) remains valid in absolute space. It is still true that every direct isometry is the product of two half-turns, and that every opposite isometry with

* W. F. Donkin, On the geometrical theory of rotation, *Philosophical Magazine* (4), **1**, (1851), 187–192. Lamb [**1**, p. 6] used half-turns about the vertices A, B, C of the given triangle to construct three new triangles which, he said, "are therefore directly equal to one another, and 'symmetrically' equal to ABC." This was a mistake: all four triangles are directly congruent!

an invariant point is a rotatory inversion (possibly reducing to a reflection or to a central inversion). Moreover, the classical enumeration of the five Platonic solids (§§ 10.1–10.3) is part of absolute geometry. The few necessary changes are easily supplied; for example, the term *rectangle* must be interpreted as meaning a quadrangle whose angles are all equal (though not necessarily right angles), and a *square* is the special case when also the sides are equal.

EXERCISES

1. If l is a line outside the plane of a triangle ABC, what can be said about the three lines in which this plane meets the three planes Al, Bl, Cl? (If two of the three lines intersect, or are parallel, or have a common perpendicular, the same can be said of all three. This property of three lines m_1, m_2, m_3 is equivalent to $R_1R_2R_3 = R_3R_2R_1$ in the notation of § 3.4.)

2. The product of reflections in the lines p and r of Figure 15.2a is a parallel displacement which transforms J into L.

15.4 FINITE GROUPS OF ROTATIONS

These groups, in particular the last three, are an immensely attractive subject for geometric investigation.

H. Weyl [**1**, p. 79]

One of the simplest kinds of transformation is a *permutation* (or rearrangement) of a finite number of named objects. For instance, one way to permute the six letters a, b, c, d, e, f is to transpose (or interchange) a and b, to change c into d, d into e, e into c, and to leave f unaltered. This permutation is denoted by $(a\ b)(c\ d\ e)$. The two "independent" parts, $(a\ b)$ and $(c\ d\ e)$, are called *cycles* of periods 2 and 3. A permutation that consists of just one cycle is said to be *cyclic*. Clearly, the cyclic group C_n may be represented by the powers of the generating permutation $(a_1a_2 \ldots a_n)$; for instance, the four elements of C_4 are

$$1, \quad (a\ b\ c\ d), \quad (a\ c)(b\ d), \quad (a\ d\ c\ b).$$

A cyclic permutation of period 2, such as $(a\ b)$, is called a *transposition.* Since

$$(a_1a_2 \ldots a_n) = (a_1a_n)(a_2a_n) \ldots,$$

any permutation may be expressed as a product of transpositions. A permutation is said to be *even* or *odd* according to the parity of the number of cycles of even period; for instance, $(a\ c)(b\ d)$ is even, but $(a\ b)(c\ d\ e)$ is odd. The identity, 1, has no cycles at all, and is accordingly classified as an even permutation. It is easily proved [see Coxeter **1**, pp. 40–41] that every product of transpositions is even or odd according to the parity of the number of transpositions. It follows that the multiplication of even and odd permutations behaves like the *addition* of even and odd numbers; for example, the product of two odd permutations is even.

It follows also that every group of permutations either consists entirely of even permutations or contains equal numbers of even and odd permutations. The group of all permutations of n objects is called the *symmetric* group of order $n!$ (or of *degree* n) and is denoted by S_n. The subgroup consisting of all the even permutations is called the *alternating* group of order $\frac{1}{2}n!$ (or of degree n) and is denoted by A_n. In particular, S_2 is the same group as C_2, and A_3 the same as C_3, so we write

$$S_2 \cong C_2, \qquad A_3 \cong C_3.$$

More interestingly, $S_3 \cong D_3$ (see Figure 2.7*a*). For, the six elements of the dihedral group D_3, being symmetry operations of an equilateral triangle, may be regarded as permutations of the three sides of the triangle. The even permutations

$$1, \quad (a\ b\ c), \quad (a\ c\ b)$$

(which form the subgroup $A_3 \cong C_3$) are rotations, whereas the odd permutations

$$(b\ c), \quad (c\ a), \quad (a\ b)$$

are reflections in the three medians. If we regard the triangle as lying in three-dimensional (absolute) space, the rotations are about an axis through the center of the triangle, perpendicular to its plane. The reflections may then be interpreted in two alternative ways, yielding two groups which are geometrically distinct but abstractly identical or *isomorphic*: we may either reflect in three planes through the axis or rotate through half-turns about the medians themselves. In the latter representation, all the six elements of D_3 appear as rotations. We may describe this as the group of direct symmetry operations of a triangular prism. More generally, the $2n$ direct symmetry operations of an n-gonal prism form the dihedral group D_n, whereas of course the n direct symmetry operations of an n-gonal pyramid form the cyclic group C_n. The rotations of C_n all have the same axis, and D_n is derived from C_n by adding half-turns about n lines symmetrically disposed in a plane perpendicular to that axis.

We have thus found two infinite families of finite groups of rotations. Other such groups are the groups of direct symmetry operations of the five Platonic solids $\{p, q\}$. These are only three groups, not five, because any rotation that takes $\{p, q\}$ into itself also takes the reciprocal $\{q, p\}$ into itself: the octahedron has the same group of rotations as the cube, and the icosahedron the same as the dodecahedron.

The regular tetrahedron $\{3, 3\}$ is evidently symmetrical by reflection in the plane that joins any edge to the midpoint of the opposite edge. As a permutation of the four faces a, b, c, d (Figure 15.4*a*), this reflection is just a transposition. Thus the complete symmetry group of the tetrahedron,

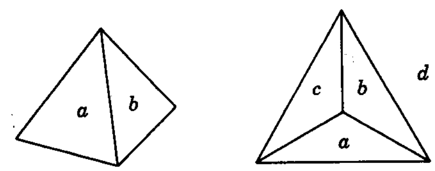

Figure 15.4a

being generated by such reflections, is isomorphic to the symmetric group S_4, which is generated by transpositions; and the rotation group, being generated by products of pairs of reflections, is isomorphic to the alternating group A_4, which is generated by products of pairs of transpositions. The 12 rotations may be counted as follows. The perpendicular from a vertex to the opposite face is the axis of a *trigonal* rotation (i.e., a rotation of period 3); the 4 vertices yield 8 such rotations. The line joining the midpoint of two opposite edges is the axis of a half-turn (or *digonal* rotation); the 3 pairs of opposite edges yield 3 such half-turns. Including the identity, we thus have $8 + 3 + 1 = 12$ rotations. As permutations, the 8 trigonal rotations are

$$(b\,c\,d),\quad (b\,d\,c),\quad (a\,c\,d),\quad (a\,d\,c),\quad (a\,b\,d),\quad (a\,d\,b),\quad (a\,b\,c),\quad (a\,c\,b)$$

and the 3 half-turns are

$$(b\,c)(a\,d),\quad (c\,a)(b\,d),\quad (a\,b)(c\,d).$$

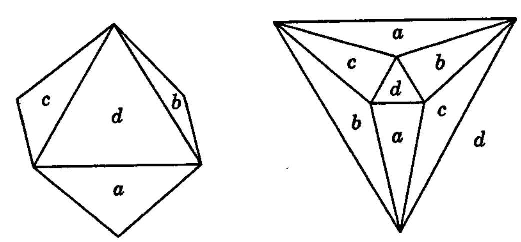

Figure 15.4b

The octahedron $\{3, 4\}$ can be derived from the tetrahedron by *truncation*: its eight faces consist of the four vertex figures of the tetrahedron and truncated versions of the four faces. Every symmetry operation of the tetrahedron is retained as a symmetry operation of the octahedron, but the octahedron also has symmetry operations that interchange the two sets of four faces. For instance, the line joining two opposite vertices is the axis of a *tetragonal* rotation (of period 4), and the line joining the midpoints of two opposite edges is the axis of a half-turn. When the four pairs of opposite faces are marked *a*, *b*, *c*, *d*, as in Figure 15.4*b*, such a half-turn appears as a transposition, which is one of the permutations that belong to S_4 but not

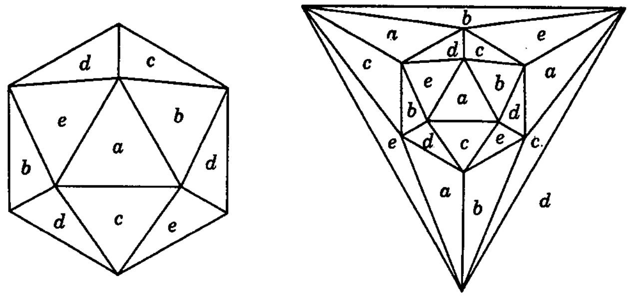

Figure 15.4c

to A_4. It follows that the rotation group of the octahedron (or of the cube) is isomorphic to the symmetric group S_4.

In Figure 15.4*c*, the twenty faces of the icosahedron {3, 5} have been marked *a*, *b*, *c*, *d*, *e* in sets of four, in such a way that two faces marked alike have nothing in common, not even a vertex. In fact, the four *a*'s (for instance) lie in the planes of the faces of a regular tetrahedron, and the respectively opposite faces (marked *b*, *c*, *d*, *e*) form the reciprocal tetrahedron. The twelve rotations of either tetrahedron into itself (represented by the even permutations of *b*, *c*, *d*, *e*) are also symmetry operations of the whole icosahedron. This behavior of the four *a*'s is imitated by the *b*'s, *c*'s, *d*'s and *e*'s, so that altogether we have all the even permutations of the five letters: the rotation group of the icosahedron (or of the dodecahedron) is isomorphic to the alternating group A_5. The 60 rotations may be counted as follows: 4 pentagonal rotations about each of 6 axes, 2 trigonal rotations about each of 10 axes, 1 half-turn about each of 15 axes, and the identity [Coxeter **1**, p. 50].

We shall find that the above list exhausts the finite groups of rotations. As a first step in this direction, we observe that all the axes of rotation must pass through a fixed point. In fact, we can just as easily prove a stronger result:

15.41 *Every finite group of isometries leaves at least one point invariant.*

Proof. A finite group of isometries transforms any given point into a finite set of points, and transforms the whole set of points into itself. This, like any finite (or bounded) set of points, determines *a unique smallest sphere that contains all the points on its surface or inside*: unique because, if there were two equal smallest spheres, the points would belong to their common part, which is a "lens"; and the sphere that has the rim of the lens for a great circle is smaller than either of the two equal spheres, contradicting our supposition that these spheres are as small as possible. (The shaded area in Figure 15.4*d* is a section of the lens.) The group transforms this unique sphere into itself. Its surface contains some of the points, and therefore all of them. Its center is the desired invariant point.

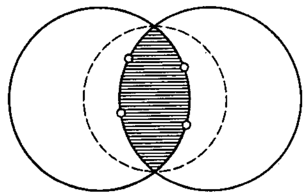

Figure 15.4d

It follows that any finite group of rotations may be regarded as operating on the surface of a sphere. In such a group G, each rotation, other than the identity, leaves just two points invariant, namely the *poles* where the axis of rotation intersects the sphere. A pole P is said to be p-gonal ($p \geqslant 2$) if it belongs to a rotation of period p. The p rotations about P, through various multiples of the angle $2\pi/p$, are those rotations of G which leave P invariant. Any other rotation of G transforms P into an "equivalent" pole, which is likewise p-gonal. Thus all the poles fall into sets of equivalent poles. All the poles in a set have the same period p, but two poles of the same period do not necessarily belong to the same set; they belong to the same set only if one is transformed into the other by a rotation that belongs to G.

Any set of equivalent p-gonal poles consists of exactly n/p poles, where n is the order of G. To prove this, take a point Q on the sphere, arbitrarily near to a pole P belonging to the set. The p rotations about P transform Q into a small p-gon round P. The other rotations of G transform this p-gon into congruent p-gons round all the other poles in the set. But the n rotations of G transform Q into just n points (including Q itself). Since these n points are distributed into p-gons round the poles, the number of poles in the set must be n/p.

The $n - 1$ rotations of G, other than the identity, consist of $p - 1$ for each p-gonal axis, that is, $\frac{1}{2}(p - 1)$ for each p-gonal pole, or

$$\tfrac{1}{2}(p - 1)n/p$$

for each set of n/p equivalent poles. Hence

$$n - 1 = \tfrac{1}{2}n \, \Sigma(p - 1)/p,$$

where the summation is over the sets of poles. This equation may be expressed as

$$2 - \frac{2}{n} = \Sigma\left(1 - \frac{1}{p}\right).$$

If $n = 1$, so that G consists of the identity alone, there are no poles, and the sum on the right has no term. In all other cases $n \geqslant 2$, and therefore

$$1 \leqslant 2 - \frac{2}{n} < 2.$$

It follows that the number of sets of poles can only be 2 or 3; for, the single term $1 - 1/p$ would be less than 1, and the sum of 4 or more terms would be

$$\geqslant 4(1 - \tfrac{1}{2}) = 2.$$

If there are 2 sets of poles, we have

$$2 - \frac{2}{n} = 1 - \frac{1}{p_1} + 1 - \frac{1}{p_2},$$

that is,

$$\frac{n}{p_1} + \frac{n}{p_2} = 2.$$

But two positive integers can have the sum 2 only if each equals 1; thus

$$p_1 = p_2 = n,$$

each of the 2 sets of poles consists of one n-gonal pole, and we have the cyclic group C_n with a pole at each end of its single axis.

Finally, in the case of 3 sets of poles we have

$$2 - \frac{2}{n} = 1 - \frac{1}{p_1} + 1 - \frac{1}{p_2} + 1 - \frac{1}{p_3},$$

whence

15.42 $$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 + \frac{2}{n}.$$

Since this is greater than $\tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3} = 1$, the three periods p_i cannot all be 3 or more. Hence at least one of them is 2, say $p_3 = 2$, and we have

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2} + \frac{2}{n},$$

whence

$$(p_1 - 2)(p_2 - 2) = 4(1 - p_1p_2/n) < 4$$

(cf. 10.33), so that the only possibilities (with $p_1 \leqslant p_2$ for convenience) are:

$$p_1 = 2,\ p_2 = p,\ n = 2p; \qquad p_1 = 3,\ p_2 = 3,\ n = 12;$$
$$p_1 = 3,\ p_2 = 4,\ n = 24; \qquad p_1 = 3,\ p_2 = 5,\ n = 60.$$

We recognize these as the dihedral, tetrahedral, octahedral and icosahedral groups.

This completes our proof [Klein **3**, p. 129] that

15.43 *The only finite groups of rotations in three dimensions are the cyclic groups C_p ($p = 1, 2, \ldots$), the dihedral groups D_p ($p = 2, 3, \ldots$), the tetrahedral group A_4, the octahedral group S_4, and the icosahedral group A_5.*

(To avoid repetition, we have excluded D_1 which, when considered as a group of rotations, is not only abstractly but geometrically identical with C_2.)

Any solid having one of these groups for its complete symmetry group

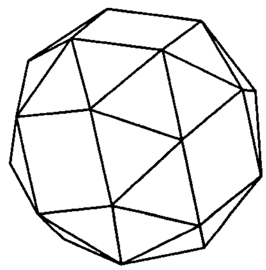

Figure 15.4e

(such as the Archimedean *snub cube** shown in Figure 15.4*e*, whose group is S_4) can occur in two *enantiomorphous* varieties, *dextro* and *laevo* (i.e., right- and left-handed): mirror images that cannot be superposed by a continuous motion.

EXERCISES

1. Interpret the following permutations as rotations of the octahedron (Figure 15.4*b*):

$$(a\,b\,c\,d),\quad (a\,b\,c),\quad (a\,b),\quad (a\,b)(c\,d).$$

Count the rotations of each type, and check with the known order of S_4.

2. Using the symbol (p_1, p_2, p_3) for the group having three sets of poles of periods p_1, p_2, p_3, consider the possibility of stretching the notation so as to allow $(1, p, p) \cong C_p$ as well as

$$(2, 2, p) \cong D_p, \qquad (2, 3, 3) \cong A_4,$$
$$(2, 3, 4) \cong S_4, \qquad (2, 3, 5) \cong A_5.$$

15.5 FINITE GROUPS OF ISOMETRIES

Having enumerated the finite groups of rotations, we can easily solve the wider problem of enumerating the finite groups of isometries (cf. § 2.7). Since every such group leaves one point invariant, we are concerned only with isometries having fixed points. Such an isometry is a rotation or a rotatory inversion according as it is direct or opposite (7.15, 7.41).

If a finite group of isometries consists entirely of rotations, it is one of the groups G considered in § 15.4. If not, it contains such a group G as a subgroup of index 2, that is, it is a group of order $2n$ consisting of n rotations $S_1, S_2, \ldots, S_n$ and an equal number of rotatory inversions T_1,

* The vertices of the snub cube constitute a distribution of 24 points on a sphere for which the smallest distance between any 2 is as great as possible. This was conjectured by K. Schütte and B. L. van der Waerden (*Mathematische Annalen,* **123** (1951), pp. 108, 123) and was proved by R. M. Robinson (*ibid.,* **144** (1961), pp. 17–48). The analogous distribution of 6 or 12 points is achieved by the vertices of an octahedron or an icosahedron, respectively. For 8 points the figure is not, as we might at first expect, a cube, but a square antiprism [Fejes Tóth **1**, pp. 162–164].

$T_2, \ldots, T_n$. For, if the group consists of n rotations S_i and (say) m rotatory inversions T_i, we can multiply by T_1 so as to express the same $n + m$ isometries as S_iT_1 and T_iT_1. The n isometries S_iT_1, being rotatory inversions, are the same as T_i (suitably rearranged if necessary), and the m isometries T_iT_1, being rotations, are the same as S_i. Therefore $m = n$.

If the central inversion I belongs to the group, the n rotatory inversions are simply

$$S_iI = IS_i \quad (i = 1, 2, \ldots, n),$$

and the group is the direct product $G \times \{I\}$, where G is the subgroup consisting of the S's and $\{I\}$ denotes the group of order 2 generated by I. (As an abstract group, $\{I\}$ is, of course, the same as C_2 or D_1.)

If I does not belong, the $2n$ transformations S_i and T_iI form a group of rotations of order $2n$ which has the same multiplication table as the given group consisting of S_i and T_i. For, if $S_iT_j = T_k$,

$$S_iT_jI = T_kI,$$

and if $T_iT_j = S_k$,

$$T_iIT_jI = T_iI^2T_j = T_iT_j = S_k.$$

In other words, a group of n rotations and n rotatory inversions, not including I, is isomorphic to a rotation group G' of order $2n$ which has a subgroup G of order n. To complete our enumeration, we merely have to seek such pairs of related rotation groups. Each pair yields a "mixed" group, say $G'G$, consisting of all the rotations in the smaller group G, along with the remaining rotations in G' each multiplied by the central inversion I. Looking back at § 15.4, we see that the possible pairs are

$$C_{2n}C_n, \quad D_nC_n, \quad D_nD_{\frac{1}{2}n}\ (n \text{ even}), \quad S_4A_4.$$

Thus we can complete Table III on p. 413.

EXERCISES

1. Determine the symmetry groups of the following figures: (a) an orthoscheme $O_0O_1O_2O_3$ (Figure 10.4*c*) with $O_0O_1 = O_2O_3$; (b) an n-gonal antiprism (n even or odd).

2. Designate in the $G'G$ notation the direct product of the group of order 3 generated by a rotation about a vertical axis and the group of order 2 generated by the reflection in a horizontal plane.

15.6 GEOMETRICAL CRYSTALLOGRAPHY

The sense in which a snail's shell winds is an inheritable character founded in its genetic constitution, as is . . . the winding of the intestinal duct in the species Homo sapiens. . . . *Also the deeper chemical constitution of our human body shows that we have a screw, a screw that is turning the same way in every one of us.* . . . A horrid manifestation of this genotypical asymmetry is a metabolic disease called phenylketonuria, leading to insanity, that man contracts when a small quantity of laevo-phenylalanine is added to his food, while the dextro- form has no such disastrous effects.*

H. Weyl [**1**, p. 30]

The discussion of symmetry groups has been phrased in such a way as to be valid not only in Euclidean space but in absolute space. However, it seems appropriate to mention the application of these ideas to the practical science of crystallography. Accordingly, in this digression the geometry is strictly Euclidean.

Crystallographers are interested in those finite groups of isometries which arise as subgroups (and factor groups) of symmetry groups of three-dimensional lattices. By § 4.5, these are the special cases in which the only rotations that occur have periods 2, 3, 4 or 6. This crystallographic restriction reduces the rotation groups to

$$C_1,\ C_2,\ C_3,\ C_4,\ C_6,\ D_2,\ D_3,\ D_4,\ D_6,\ A_4,\ S_4,$$

the direct products to these eleven each multipled by $\{I\}$, and the mixed groups to

$$C_2C_1,\ C_4C_2,\ C_6C_3,\ D_2C_2,\ D_3C_3,\ D_4C_4,\ D_6C_6,\ D_4D_2,\ D_6D_3,\ S_4A_4.$$

(Of course, $C_1 \times \{I\}$ is just $\{I\}$ itself.)

These 32 groups are called the *crystallographic point groups* or "*crystal classes.*" Every crystal has one of them for its symmetry group, and every group except C_6C_3 occurs in at least one known mineral. In the more familiar notation of Schoenflies [see, e.g., Burckhardt **1**, p. 71], the groups are respectively

$$C_1,\ C_2,\ C_3,\ C_4,\ C_6,\ D_2,\ D_3,\ D_4,\ D_6,\ T,\ O,$$
$$C_i,\ C_{2h},\ C_{3i},\ C_{4h},\ C_{6h},\ D_{2h},\ D_{3d},\ D_{4h},\ D_{6h},\ T_h,\ O_h,$$
$$C_s,\ S_4,\ C_{3h},\ C_{2v},\ C_{3v},\ C_{4v},\ C_{6v},\ D_{2d},\ D_{3h},\ T_d.$$

To avoid possible confusion, observe that our C_4C_2 and S_4 ("S" for "symmetric") are Schoenflies's S_4 and O (for "octahedral"). The 32 groups are customarily divided into seven *crystal systems,* as follows:

Triclinic:	C_1, $\{I\}$.	
Monoclinic:	C_2, $C_2 \times \{I\}$, C_2C_1.	
Orthorhombic:		D_2, $D_2 \times \{I\}$, D_2C_2.

* The DNA molecule?

Rhombohedral:	C_3,	$C_3 \times \{I\}$,		D_3,	$D_3 \times \{I\}$,	D_3C_3.	
Tetragonal:	C_4,	$C_4 \times \{I\}$,	C_4C_2,	D_4,	$D_4 \times \{I\}$,	D_4C_4,	D_4D_2.
Hexagonal:	C_6,	$C_6 \times \{I\}$,	C_6C_3,	D_6,	$D_6 \times \{I\}$,	D_6C_6,	D_6D_3.
Cubic:	A_4,	$A_4 \times \{I\}$,		S_4,	$S_4 \times \{I\}$,	S_4A_4.	

Table I (on p. 413) is a complete list of the 17 discrete groups of isometries in two dimensions involving two independent translations. The analogous groups in three dimensions are the discrete groups of isometries involving three independent translations. The enumeration of these *space groups* is the central problem of mathematical crystallography. The complete list contains $65 + 165 = 230$ groups.

The first 65 are composed entirely of *direct* isometries. Although these were enumerated as long ago as 1869 by C. Jordan [see Hilton **1**, p. 258], they are usually attributed to L. Sohncke who, in 1879, pointed out their application to crystallography. The most obvious group consists of translations alone. The remaining 64 of the 65 contain also rotations and screw displacements; 22 of them occur in 11 enantiomorphous pairs which are mirror images of each other (one containing right-handed screw displacements and the other the reflected left-handed screw displacements). This explains the phenomenon of optical activity [Sayers and Eustace **1**, pp. 238–241, 248–252]. From the standpoint of pure geometry or pure group theory, it would be more natural to ignore this distinction of sense, thus reducing the number 65 to 54, and the total of 230 to 219 [Burckhardt **1**, p. 161].

The remaining 165 groups contain not only direct but also *opposite* isometries: reflections, rotatory reflections (or rotatory inversions), and glide reflections. Their enumeration, by Fedorov in Russia (1890), Schoenflies in Germany (1891), and Barlow in England (1894), provides one of the most striking instances of independent discovery in different places using different methods. Fedorov, who obtained the 230 as $73 + 54 + 103$ instead of $65 + 165$, was probably unaware of the preliminary work of Jordan and Sohncke. It is quite certain that Schoenflies knew nothing of Fedorov, and that Barlow's work was independent of both.

EXERCISE

Determine the symmetry groups of the following figures: (a) a rectangular parallelepiped (e.g., a brick), (b) a rhombohedron; (c) a regular dodecahedron with an inscribed cube (whose 8 vertices occur among the 20 vertices of the dodecahedron).

15.7 THE POLYHEDRAL KALEIDOSCOPE

In combining three reflections . . . the effect is highly pleasing.

Sir David Brewster (1781 -1848)

[Brewster **1**, p. 93]

Table III (on p. 413) is a complete list of the finite groups of isometries. In the preceding section, we selected from this list those groups which satisfy