FINAL EXAM TOPICS

GROUPS AND THEIR CONNECTIONS TO GEOMETRY

The list below contains (i) concepts, notions, examples you must be familiar and comfortable with (including some basic properties of those), and (ii) results that will appear (possibly in a somewhat modified form) as "theoretical" items in the final examination questions. "No proof" refers only to the statement in the preceding sentence. Items in the list are of unequal size and are in no direct connection with actual ticket questions.

1. Symmetric groups, alternating groups. S_n is generated by transpositions. Conjugacy classes in S_n. A_5 has no non-trivial normal subgroups.

2. Abelian groups. Classification of finitely generated abelian groups without torsion. General classification of finitely generated abelian groups (no proof). Examples of countable abelian groups with and without torsion that are not finitely generated.

3. Classification of isometries in $\mathbb{R}^2$ and $\mathbb{R}^3$. Isometries are determined by the images of three points in $\mathbb{R}^2$, four points in $\mathbb{R}^3$. Isometries are products of reflections. Every element of $SO(3)$ is a rotation. Conjugacy classes in $\mathrm{Isom}(\mathbb{R}^2)$ and $\mathrm{Isom}(\mathbb{R}^3)$.

4. Finite subgroups of $\mathrm{Isom}(\mathbb{R}^2)$ and $\mathrm{Isom}(\mathbb{R}^3)$. Discrete subgroups of $\mathrm{Isom}(\mathbb{R}^2)$. Crystallographic restriction. Factor tori and their isometries.

5. Linear representations of $\mathrm{Isom}(\mathbb{R}^2)$, $\mathrm{Isom}(\mathbb{E}^2)$ and $\mathrm{Isom}(\mathbb{H}^2)$. Linear representations of symmetries of Riemann sphere, (real) projective plane, and affine plane. $\mathrm{Aff}(\mathbb{R})$, $\mathrm{Isom}(\mathbb{R}^2)$ and $\mathrm{Aff}(\mathbb{R}^2)$ as semi-direct products.

6. Classification of fractional linear transformations of Riemann sphere (complex coefficients) and hyperbolic plane (real coefficients). Geometric description of elliptic, parabolic, and hyperbolic transformations of $\mathbb{H}^2$.

7. Synthetic description of group of symmetry transformations for Riemann sphere, $\mathbb{R}P(2)$, affine plane, $\mathbb{R}^2$, $\mathbb{E}^2$, $\mathbb{H}^2$. Preservation of distances, lines, angles, lines and circles, conic sections. Theorem: any transformation of $\mathbb{R}^2$ that maps lines to lines is affine. Similar theorems for projective plane and Riemann sphere. Preservations of angles *locally* does not imply that transformation is fractional linear.

8. Definition of scalar/Hermitian products, Hermitian/unitary/normal matrices. Normal matrices are unitarily diagonalisable over $\mathbb{C}$. Normal form of symmetric, orthogonal and normal matrices over $\mathbb{R}$.

9. Examples of Lie groups. Definition of the matrix exponential and relationship between Lie groups and Lie algebras. Nilpotent case.

10. Definition of fundamental group, independence from choice of base point. Fundamental group of circle, sphere, bouquet of circles, torus. Fundamental group of a topological group is abelian.

11. Definition of covering maps and covering spaces. Lifting paths and homotopies to covering spaces. $\mathbb{R}^2$ as a covering space of the torus. Every isometry of the torus $\mathbb{R}^2/L$ comes from an isometry of $\mathbb{R}^2$.

12. Cayley graph of a group. Examples: free groups, free abelian groups, surface groups (standard generators). Every tree is contractible. Every graph is homotopic to a bouquet of circles. Every subgroup of a free group is free.

13. Planar models and polygonal complexes for various surfaces. Relationship between finite presentation of a group and definition of a polygonal complex (generators and edges, relations and faces; no proof). Given a finitely presented group G, construction of a topological space X with $\pi_1(X) = G$.

14. Relationship between fundamental group of a space and fundamental group of a covering space. If G acts freely and discretely on a simply connected space X, then $\pi_1(X/G) = G$. The example of the surface of genus two via an octagon in the hyperbolic plane (no proof).

15. Definition of a Fuchsian group and a fundamental domain. Examples of Fuchsian groups with fundamental domains that are (1) bounded, (2) unbounded but finite area, (3) unbounded and infinite area.

16. Embedding F_2 into $\mathrm{Isom}(\mathbb{H}^2)$. There are subgroups of F_2 isomorphic to F_n for every n, including $n = \infty$. Embedding F_n into $\mathrm{Isom}(\mathbb{H}^2)$.

17. Fundamental group of the figure-eight. A covering space of the figure-eight with trivial fundamental group. A covering space with fundamental group F_∞.

18. A presentation of $PSL(2,\mathbb{Z})$ in which all generators have infinite order. A presentation in which all generators have finite order.

19. Definition of commensurable groups. Definition of exponential and polynomial growth. Degree of polynomial growth does not depend on choice of generators. Presence of exponential growth does not depend on generators, but rate of growth does.

20. Relationship of following categories of groups: abelian, free, nilpotent, solvable, surface groups. Growth rates of groups in these categories.

21. Definition of the outer automorphism group. Construction of $GL(n,\mathbb{Z})$ as $\mathrm{Out}(\mathbb{Z}^n)$. Correspondence between $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ and space of lattices in $\mathbb{R}^n$.

22. $SL(n,\mathbb{Z})$ is generated by elementary matrices. Generating relations (no proof). System of generators with a fixed number of elements.

23. Semi-direct products. Semi-direct product of solvable groups is solvable. Example of semi-direct product of abelian groups that is not nilpotent. Not finitely generated Abelian subgroup of a finitely generated solvable group.

24. Quasi-isometries between metric spaces. Quasi-isometry between a group and its Cayley graph. Invariance of exponential and polynomial growth under quasi-isometry.