MASS algebra

Midterm solutions

1.1

Let G be a gp of order 10. Every element of G has order 1, 2, 5, or 10. If some $g \in G$ has order 10, then $G = \langle g \rangle \cong \mathbb{Z}/10\mathbb{Z}$ is abelian. Similarly, if <u>every</u> $g \in G$ has order 2, we know G is abelian. Thus it only remains to consider the case when G has an element of order 5 & no elements of order 10.

Let $a \in G$ have order 5 and consider $b \in G \setminus \langle a \rangle$. Because $\langle a \rangle$ has index 2, we have $[b\langle a \rangle]^2 = b^2\langle a \rangle = \langle a \rangle$, hence $b^2 \in \langle a \rangle$. Observe that a, a^2, a^3, a^4 all have order 5, while b has order 2 or 5 (not 10) $\therefore$ b^2 cannot be any of these. Thus $b^2 = e$.

By the same argument, we have $g^2 = e$ $\forall g \in b\langle a \rangle$. In particular, $baba = e$, and so $ba = a^4 b$. Thus $G = \langle a, b \mid a^5 = b^2 = e,\ ba = a^4 b \rangle$ is isomorphic to the dihedral group D_5.

$\boxed{1.2}$

(1) $\mathrm{Isom}^+(\mathbb{R}^3)$ embeds into $GL(4,\mathbb{R})$ as

$$\begin{bmatrix} SO(3) & \mathbb{R}^3 \\ \vec{0} & 1 \end{bmatrix} \qquad \left(\begin{matrix} A \in SO(3), \ \vec{b} \in \mathbb{R}^3 \\ \longleftrightarrow \quad \vec{x} \longmapsto A\vec{x} + \vec{b} \end{matrix} \right)$$

Let $G = \{ T \in \mathrm{Isom}^+(\mathbb{R}^3) \mid TP = P \}$, where P is the xy-plane. That is, an isometry $T: \vec{x} \longmapsto A\vec{x} + \vec{b} = \vec{y}$ is in G if and only if $y_3 = 0$ whenever $x_3 = 0$.
Observe that $y_3 = A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + b_3$.
Thus we require A_{31}, A_{32}, and b_3 to satisfy $A_{31}x_1 + A_{32}x_2 + b_3 = 0 \quad \forall\, x_1, x_2 \in \mathbb{R}$. This is possible if and only if $A_{31} = A_{32} = b_3 = 0$. Thus

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}$$

Furthermore, $A \in SO(3) \Rightarrow$ the columns of A form an o.n. basis of $\mathbb{R}^3$. The first two columns span P $\therefore$ the third lies in $P^\perp = \{(0,0,z) \mid z \in \mathbb{R}\}$. It follows that

$$A = \left[\begin{array}{c|c} \overset{2\times 2}{A'} & \overset{2\times 1}{\vec{0}} \\ \hline \underset{1\times 2}{\vec{0}} & \underset{1\times 1}{\pm 1} \end{array}\right] \qquad \text{where } \det A' = A_{33}.$$

Every $A, \vec{b}$ of the given form gives an even isometry of $\mathbb{R}^3$ preserving P, and so G embeds into $GL(4,\mathbb{R})$ as

$$\left\{ \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{A'} & 0 & b_1 \\ & & 0 & b_2 \\ \hline 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] : \begin{matrix} A' \in O(2), & \sigma = \pm 1, \\ \det A' = \sigma, & b_1, b_2 \in \mathbb{R} \end{matrix} \right\}$$

Even more explicitly, this is

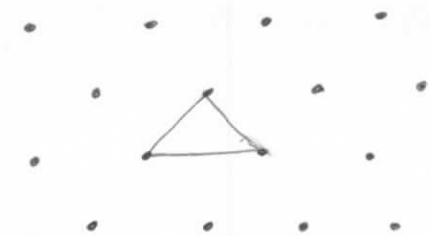
$$(*) \quad \left\{ \begin{bmatrix} \cos\theta & -\sigma\sin\theta & 0 & b_1 \\ \sin\theta & \sigma\cos\theta & 0 & b_2 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : \begin{matrix} \theta \in [0, 2\pi), \\ \sigma = \pm 1, \\ b_1, b_2 \in \mathbb{R} \end{matrix} \right\}$$

If the plane P is something other than the xy-plane a suitable isometric change of coordinates will put G in the above form, once again yielding the matrix representation (*).

(2) $\mathrm{Aff}(\mathbb{R}^3)$ embeds into $GL(4,\mathbb{R})$ as $\begin{bmatrix} GL(3,\mathbb{R}) & \mathbb{R}^3 \\ \vec{0} & 1 \end{bmatrix}$.

Once again, $T: \vec{x} \mapsto A\vec{x} + \vec{b}$ preserves P iff $A_{31} = A_{32} = b_3 = 0$, but now A_{13}, A_{23} may be arbitrary. We require $A_{33} \neq 0$ since $\det A \neq 0$, but there are no other restrictions, and so the group of affine maps preserving P embeds as

$$\left\{ \begin{bmatrix} A' & & c_1 & b_1 \\ & & c_2 & b_2 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : \begin{array}{l} A' \in GL(2,\mathbb{R}), \\ \sigma \in \mathbb{R} \setminus \{0\}, \\ \vec{b}, \vec{c} \in \mathbb{R}^2 \end{array} \right\}$$



$\boxed{1.3}$ The lattice L is shown at right:
Individual "cells" are equilateral triangles.
As with the usual torus $\mathbb{R}^2/\mathbb{Z}^2$, a simple lifting argument shows that every isometry of $\mathbb{R}^2/L$ is of the form $[\vec{x}] \mapsto [T\vec{x}]$, where $I \in \mathrm{Isom}(\mathbb{R}^2)$. Thus it suffices to determine which elements of $\mathrm{Isom}(\mathbb{R}^2)$ respect equivalence classes — that is, for which I we have

$$I[\vec{x}] = [I\vec{x}] \qquad \forall\ \vec{x} \in \mathbb{R}, \qquad (*)$$

where $[\vec{x}] = \vec{x} + L = \{ \vec{x} + a(1,0) + b(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) : a, b \in \mathbb{Z}^2 \}$

Translations| Given $\vec{v} \in \mathbb{R}^2$ & $I : \vec{x} \mapsto \vec{x} + \vec{v}$, we have

$$I[\vec{x}] = I(\vec{x} + L) = \vec{x} + L + \vec{v} = [\vec{x} + \vec{v}] = [I\vec{x}]$$

$\therefore$ <u>every</u> translation of $\mathbb{R}^2$ gives an isometry of $\mathbb{R}^2/L$.

Rotations| Let I be rotation by θ around a centre $\vec{p}$.
Consider $\vec{x} = \vec{p} + (1,0) \in [\vec{p}]$, and observe that since $\vec{p}$ is fixed, we have $[I\vec{p}] = [\vec{p}]$, thus $(*) \Rightarrow$

$$[I\vec{x}] = I[\vec{x}] = I[\vec{p}] = [I\vec{p}] = [\vec{p}]$$

$\therefore\ I\vec{x} \in [\vec{p}]$. There are exactly 6 elements of $[\vec{p}] = \vec{p} + L$ lying on the circle of radius 1 around $\vec{p}$, and these correspond to rotations by a multiple of $\frac{\pi}{3}$.
Thus $\theta = \frac{k\pi}{3}$ for $k = 0, 1, 2, 3, 4, 5$.

Conversely, given <u>any</u> $\vec{p} \in \mathbb{R}^2$ and $\theta = \frac{\pi}{3}$, consider an arbitrary $\vec{x} \in \mathbb{R}^2$. Given $\vec{y} \in [\vec{x}]$, let $a, b \in \mathbb{Z}$ be such that $\vec{y} = \vec{x} + a(1,0) + b(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2})$. I is the rotation around $\vec{p}$ by $\frac{\pi}{3}$, and can be written in the form $I : \vec{x} \mapsto A\vec{x} + \vec{c}$, where $\vec{c} \in \mathbb{R}^2$ & A is rotation by $\frac{\pi}{3}$ around $\vec{0}$ (a linear map).
Then

$$I\vec{y} = A\vec{y} + \vec{c} = A\vec{x} + a(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) + b(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) + \vec{c}$$
$$= I\vec{x} + (a+b)(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) - b(1,0),$$

and it follows that $I[\vec{x}] = [I\vec{x}]$, so I is an isometry of the torus $\mathbb{R}^2/L$. Furthermore, $(*)$ also holds for I^2, I^3, etc., and thus every rotation by a multiple of $\frac{\pi}{3}$ (around any centre) is an isometry of $\mathbb{R}^2/L$.

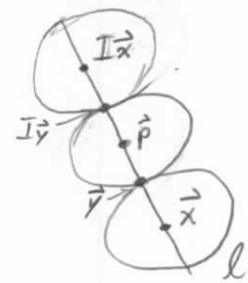
<u>Reflections</u> Let ℓ be a line which makes an angle of $\frac{k\pi}{6}$ with the horizontal direction, $k = 0, 1, 2, 3, 4, 5$, and let I be reflection in ℓ. If ℓ passes through $\vec{0}$, then one easily sees that $I(1,0)$ and $I(\frac{1}{2}, \frac{\sqrt{3}}{2})$ are on the lattice L, and so the same argument as in the last section shows that I satisfies $(*)$ and is an isometry of $\mathbb{R}^2/L$.

Conversely, if ℓ makes any other angle with the x-axis, then composing I with reflection in the x-axis yields a rotation by θ, where θ is not a multiple of $\frac{\pi}{3}$.

Thus a reflection in ℓ induces an isometry of $\mathbb{R}^2/L$ iff ℓ makes an angle of $\frac{k\pi}{6}$ with the horizontal (the location of ℓ is irrelevant).

<u>Glide reflections</u> Let ℓ be any line as in the previous section, and let I be a glide reflection along ℓ. Then I is an isometry of $\mathbb{R}^2/L$ (being the composition of two such isometries). Conversely, if I is glide reflection along any other line, then $I = T \circ J$, where J is a forbidden reflection and T is translation. If $I \in Isom(\mathbb{R}^2/L)$, then $J = T^{-1} \circ I$ is as well, and so I is not an isometry of $\mathbb{R}^2/L$.

$\boxed{1,4}$ Let $I \in Isom^+(\mathbb{R}^n)$, and suppose $\vec{x} \in \mathbb{R}^n$ is a non-fixed point of period two. Let ℓ be the line through $\vec{x}$ and $I\vec{x}$, and let $\vec{p}$ be the midpoint of $\vec{x}$ and $I\vec{x}$. That is,



$d(\vec{p}, \vec{x}) = d(\vec{p}, I\vec{x})$. Since I is an isometry, $d(I\vec{p}, I\vec{x}) = d(\vec{p}, \vec{x})$, and also

$$d(I\vec{p}, \vec{x}) = d(I\vec{p}, I^2x) = d(\vec{p}, I\vec{x})$$

(*) Thus $I\vec{p} = \vec{p}$. Now given any $\vec{y} \in \ell \setminus \{\vec{p}\}$, we have

① $d(I\vec{y}, \vec{p}) = d(I\vec{y}, I\vec{p}) = d(\vec{y}, \vec{p})$

② $d(I\vec{y}, I\vec{x}) = d(\vec{y}, \vec{x})$.

The circles of radius $d(\vec{y}, \vec{p})$ & $d(\vec{y}, \vec{x})$ centred at $\vec{p}$ & $\vec{x}$, respectively, meet in a single point, $\vec{y}$, and the same is true of the circles with these radii centred at $\vec{p}$ & $I\vec{x}$, so this intersection must be $I\vec{y}$. A similar argument shows that $I^2\vec{y} = \vec{y}$; hence I acts on ℓ as reflection in $\vec{p}$, and we are done.

(*) This is because the circles of radius $d(\vec{p}, \vec{x})$ centred at $\vec{x}$ and $I\vec{x}$ meet in just one point, $\vec{p}$.

2.1 <u>Translations</u>: Given $r \in \mathbb{R}^+$, the set of all translations by a vector of length r is a conjugacy class.

<u>Rotations</u>: Given $0 < \theta \leq \pi$, the set of all rotations by θ (around any axis) is a conjugacy class.

<u>Screw motions</u>: Given $0 < \theta \leq \pi$ and $r \in \mathbb{R}^+$, the set of all screw motions $T_{\vec{v}} \circ R^{\ell}_{\pm\theta}$, where $\ell \parallel \vec{v}$, $\|\vec{v}\| = r$, is a conjugacy class.

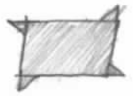
<u>Reflections</u>: The set of all reflections is a conjugacy class.

<u>Glide reflections</u>: Given $r \in \mathbb{R}^+$, the set of all glide reflections whose translation part is by a vector of length r is a conjugacy class.

<u>Rotatory reflections</u>: Given $0 < \theta \leq \pi$, the set of all rotatory reflections whose rotation part is by an angle θ is a conjugacy class.

<u>Identity</u>: The identity map is its own conjugacy class.

2.2

C_n – even isometries of a marked n-gon,
such as , lying in a plane in $\mathbb{R}^3$

$C_n \times (\mathbb{Z}/2\mathbb{Z})$ – all isometries of a marked n-gon, including reflection in the plane containing it.

D_n – even isometries of a regular n-gon

$D_n \times (\mathbb{Z}/2\mathbb{Z})$ – all isometries of a regular n-gon

A_4 – even isometries of a regular tetrahedron

$A_4 \times (\mathbb{Z}/2\mathbb{Z})$ – A_4 together with the central symmetry.

S_4 – all isometries of a regular tetrahedron / even isometries of a cube/octahedron

$S_4 \times (\mathbb{Z}/2\mathbb{Z})$ – all isometries of a cube/octahedron

A_5 – even isometries of a dodecahedron/icosahedron

$A_5 \times (\mathbb{Z}/2\mathbb{Z})$ – all isometries of a dodecahedron/icosahedron.

$\boxed{3.1}$ Count by cycle structure. Order 3:

$(\cdot\ \cdot\ \cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot) \to \binom{9}{3}\cdot 2$ ← different orderings of the 3-cycle

$(\cdot\ \cdot\ \cdot)(\cdot\ \cdot\ \cdot)(\cdot)(\cdot)(\cdot) \longrightarrow \binom{9}{3}\cdot 2\cdot\binom{6}{3}\cdot 2\cdot\frac{1}{2}$

orders in which 3-cycles may be written

$(\cdot\ \cdot\ \cdot)(\cdot\ \cdot\ \cdot)(\cdot\ \cdot\ \cdot) \longrightarrow \binom{9}{3}\cdot 2\cdot\binom{6}{3}\cdot 2\cdot\binom{3}{3}\cdot 2\cdot\frac{1}{3!}$

$$2\binom{9}{3}\left[1+\binom{6}{3}+\binom{6}{3}\cdot\frac{2}{3}\right] = \frac{9\cdot 8\cdot 7}{3}\left[1+\frac{6\cdot 5\cdot 4}{3\cdot 2}+\frac{6\cdot 5\cdot 4}{3\cdot 2}\cdot\frac{2}{3}\right]$$

$$= 3\cdot 56\left(1+20+20\cdot\tfrac{2}{3}\right) = 56(3+60+40)$$

$$= 56(103) = \boxed{5768}$$

All even (3-cycles are even)

Order 4:

ODD $(\cdot\ \cdot\ \cdot\ \cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot) \to \binom{9}{4}\cdot 3!$

EVEN $(\cdot\ \cdot\ \cdot\ \cdot)(\cdot\ \cdot)(\cdot)(\cdot)(\cdot) \to \binom{9}{4}\cdot 3!\cdot\binom{5}{2}$

ODD $(\cdot\ \cdot\ \cdot\ \cdot)(\cdot\ \cdot)(\cdot\ \cdot)(\cdot) \to \binom{9}{4}\cdot 3!\cdot\binom{5}{2}\binom{3}{2}\cdot\frac{1}{2}$ ← different orders of 2-cycles

EVEN $(\cdot\ \cdot\ \cdot\ \cdot)(\cdot\ \cdot\ \cdot\ \cdot)(\cdot) \to \binom{9}{4}\cdot 3!\cdot\binom{5}{4}\cdot 3!\cdot\frac{1}{2}$

$$\text{Total} = \binom{9}{4}\cdot 6\cdot\left[1+\binom{5}{2}+\binom{5}{2}\frac{3}{2}+\binom{5}{4}\cdot 3\right]$$

$$= \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2}\cdot 6\,[1+30+45+15]$$

$$= 9\cdot 2\cdot 7\cdot 6\,[91] = \boxed{68{,}796}$$

$$\text{Even} = \binom{9}{4}\cdot 6\cdot\left[\binom{5}{2}+\binom{5}{4}\cdot 3\right] = 9\cdot 2\cdot 7\cdot 6\,(45) = \boxed{34{,}020}$$

Order 5:

$(\cdot\ \cdot\ \cdot\ \cdot\ \cdot)(\cdot)(\cdot)(\cdot)(\cdot) \to \binom{9}{5}\cdot 4!$

$$\frac{9\cdot 8\cdot 7\cdot 6\cdot 5}{5\cdot 4\cdot 3\cdot 2\cdot 1}\cdot 4! = 9\cdot 8\cdot 7\cdot 6 = \boxed{3{,}024}$$

All even

For the sake of completeness:

Order 2

( )	$\binom{9}{2}$
( )( )	$\binom{9}{2}\binom{7}{2}\cdot\frac{1}{2}$
( )( )( )	$\binom{9}{2}\binom{7}{2}\binom{5}{2}\cdot\frac{1}{6}$
( )( )( )( )	$\binom{9}{2}\binom{7}{2}\binom{5}{2}\binom{3}{2}\cdot\frac{1}{24}$

$$\text{Total} = \binom{9}{2}\left[1 + \frac{7\cdot6}{4} + \frac{7\cdot6}{2}\cdot\frac{5\cdot4}{2}\cdot\frac{1}{6} + \frac{7\cdot6}{2}\cdot\frac{5\cdot4}{2}\cdot\frac{3\cdot2}{2}\cdot\frac{1}{24}\right]$$

$$= 36\left[1 + \frac{21}{2} + 35 + \frac{7\cdot5\cdot3}{2\cdot2}\right]$$

$$= 9\left[4 + 42 + 140 + 105\right] = 9(291) = \boxed{2619}$$

Order 6

6 + 1 + 1 + 1	$\binom{9}{6}5!$
6 + 2 + 1	$\binom{9}{6}5!\cdot\binom{3}{2}$
6 + 3	$\binom{9}{6}5!\cdot\binom{3}{3}\cdot2$

$$\binom{9}{6}5!\left[1+\binom{3}{2}+2\right] = \frac{9\cdot8\cdot7}{3!}\cdot5!\left[6\right]$$

$$= 9\cdot8\cdot7\cdot5! = \boxed{60{,}480}$$

Order 7 $\quad 7+1+1: \binom{9}{7}6! = \frac{9\cdot8}{2\cdot1}\cdot6! = 9\cdot8\cdot6\cdot5\cdot4\cdot3 = \boxed{25{,}920}$

Order 8 $\quad 8+1: \binom{9}{8}\cdot7! = 9\cdot7! = \boxed{45{,}360}$

Order 9 $\quad 8! = \boxed{40{,}320}$

Order 10 $\quad 5+2+1+1: \binom{9}{5}\cdot4!\cdot\binom{4}{2}$

$5+2+2: \quad \binom{9}{5}\cdot4!\cdot\binom{4}{2}\cdot\binom{2}{2}\cdot\frac{1}{2}$

$\rightarrow 9\cdot8\cdot7\cdot6\,(6+3) = \boxed{27{,}216}$

Order 12 $\quad 4+3+1+1: \binom{9}{4}\cdot3!\cdot\binom{5}{3}\cdot2 = \frac{9\cdot8\cdot7\cdot6}{4}\cdot5\cdot4$

$= 9\cdot8\cdot7\cdot6\cdot5 = \boxed{15{,}120}$

Order 14 $\quad 7+2: \binom{9}{7}\cdot6!\cdot\binom{2}{2} = \boxed{25{,}920}$

Order 15 $\quad 5+3+1: \binom{9}{5}\cdot4!\cdot\binom{4}{3}\cdot2! = 3024\cdot8 = \boxed{24{,}192}$

Order 20 $\quad 5+4: \binom{9}{5}\cdot4!\cdot\binom{4}{4}\cdot3! = 3024\cdot6 = \boxed{18{,}144}$

So in all, we have:

Order	# of elements in S_9
1	1
2	2619
3	5768
4	68796
5	3024
6	60480
7	25920
8	45360
9	40320
10	27216
12	15120
14	25920
15	24192
20	18144

$\boxed{362880} = 9! = |S_9|$

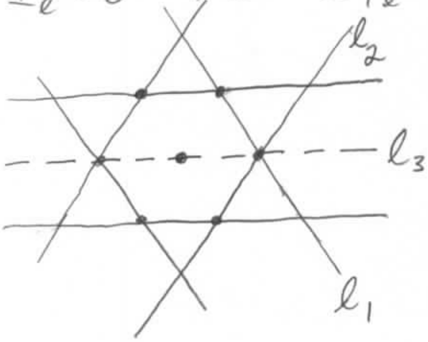
$\boxed{3.2}$ $D_6 = \langle a, b \mid a^6 = b^2 = e, \ ba = a^5 b \rangle$

$A_4 = \{ \sigma \in S_4 \mid \sigma \text{ even} \}$

There is also the following group:

$$\langle a, b \mid a^4 = b^3 = e, \quad ba = ab^2 \rangle$$

$\boxed{3.3}$ First we describe G by characterising which reflections it contains. Given a line ℓ, let I_ℓ denote reflection in ℓ, and observe that for $T \in \mathrm{Isom}(\mathbb{R}^3)$, $T \circ I_\ell \circ T^{-1} = I_{T\ell}$. In particular, if $T \in G$ and $I_\ell \in G$ then $I_{T\ell} \in G$ as well.



Observe that $I_{\ell_2}(\ell_1) = \ell_3$ $\therefore I_{\ell_3} \in G$. Similarly, $I_\ell \in G$ for every line ℓ with slope $\pm\sqrt{3}$ or 0 through one of the 7 points shown. Let L be the lattice $\mathbb{Z}(1,0) + \mathbb{Z}(\frac{1}{2}, \frac{\sqrt{3}}{2})$ from 1.3 — then given any line ℓ with slope $\in \{0, \pm\sqrt{3}\}$ passing through a point in L, the reflection I_ℓ is in G. These reflections generate G.

G comprises four classes of isometries:

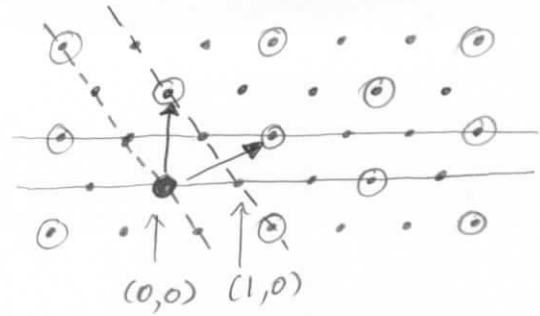
① The reflections I_ℓ just described.

② Rotations by $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ around points in L.

③ Translations by vectors of the form
$a(0, \sqrt{3}) + b(\frac{3}{2}, \frac{\sqrt{3}}{2})$, $a, b \in \mathbb{Z}$

④ Glide reflections corresponding to these translations,

L is as shown:



Reflections in the parallel pairs of lines displayed generate the translation subgroup of G. The circled lattice points are the image of $(0,0)$ under elements of G. Observe that L decomposes into 3 disjoint orbits under the action of G.

Now we describe the conjugacy classes of G.

① All reflections are conjugate since G acts transitively on the corresponding lines.

② Each orbit of G in L determines a conjugacy class containing all rotations in G whose centre is in the orbit.

③ Two translations are conjugate iff the corresponding vectors have the same length

④ The same is true of glide reflections.

To find the normal subgroups, we need unions of conjugacy classes which are closed under composition. Observe that no non-trivial normal subgroup contains a reflection. Furthermore, <u>every</u> non-trivial normal subgroup contains the translation subgroup

$\mathcal{T}_{L'} = \{ T_{\vec{v}} \mid \vec{v} \in L' \}$, where L' is some

sublattice of L, as follows.

Let $H \subset G$ be a normal subgroup, and suppose $R^{\vec{x}} = R^{\vec{x}}_{\frac{2\pi}{3}} \in H$ for some $\vec{x} \in L$. Then $R^{\vec{y}} \in H \;\; \forall \;\; \vec{y} \in G\vec{x} \subset L$, and in particular, $R^{\vec{y}}_{\frac{4\pi}{3}} \circ R^{\vec{x}}_{\frac{2\pi}{3}} = T_{(\frac{3}{2}, \frac{3}{2}\sqrt{3})}$,

while $R^{\vec{y}}_{\frac{2\pi}{3}} \circ R^{\vec{x}}_{\frac{4\pi}{3}} = T_{(3,0)}$, so

$H \supset \mathcal{T}_{L'}$, where $L' = \mathbb{Z}(3,0) + \mathbb{Z}(\frac{3}{2}, \frac{3}{2}\sqrt{3})$.

Observe that if H contains a translation <u>not</u> in L', then H contains all translations in G — that is, $H \supset \mathcal{T}_{\tilde{L}}$, where $\tilde{L} = \mathbb{Z}(0, \sqrt{3}) + \mathbb{Z}(\frac{3}{2}, \frac{\sqrt{3}}{2})$.

Let L_1, L_2, L_3 be the 3 orbits of G in L, and let $\mathcal{R}_i$ be the set of all rotations centred in L_i. Then we have the following normal subgroups of G:

$$\mathcal{R}_1 \cup \mathcal{T}_{L'}, \qquad \mathcal{R}_2 \cup \mathcal{T}_{L'} \qquad \mathcal{R}_3 \cup \mathcal{T}_{L'}$$

$$\mathcal{R}_1 \cup \mathcal{T}_{\tilde{L}} \qquad \mathcal{R}_2 \cup \mathcal{T}_{\tilde{L}} \qquad \mathcal{R}_3 \cup \mathcal{T}_{\tilde{L}}$$

$$\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{T}_{\tilde{L}} = G \cap \mathrm{Isom}^+(\mathbb{R}^2)$$

These are all the normal subgroups containing rotations. A subgroup of $\mathcal{T}_{\tilde{L}}$ corresponds to a sublattice of $\tilde{L}$. Normality implies this sublattice is of the form $k\tilde{L}$ or kL' for some $k \in \mathbb{N}$, and so we also have the normal subgroups

$$\mathcal{T}_{k\tilde{L}} \qquad \mathcal{T}_{kL'} \qquad k \in \mathbb{N}$$

Finally, given $E \subset \mathbb{R}^2$, let $\mathcal{G}_E$ be the set of glide reflections whose translation vector lies in E. The remaining normal subgroups are of the form

$$\mathcal{T}_{2k\tilde{L}} \cup \mathcal{G}_{k\tilde{L} \setminus 2k\tilde{L}} \qquad \mathcal{T}_{2kL'} \cup \mathcal{G}_{kL' \setminus 2kL'}$$