PROOF. Let $a = \int_X f\,d\mu/\mu(X)$. By convexity the graph of g lies entirely above some line through $g(a)$ (the tangent line if it exists), that is, there exists $b \in \mathbb{R}$ ($b = g'(a)$ if it exists) such that

$$g(f(x)) \geq b \cdot (f(x) - a) + g(a).$$

Now integrate both sides. □

6. Basic topology

a. Topological spaces.

1. *Topologies, bases, convergence.*

DEFINITION A.6.1. A *topological space* $(X, \mathcal{T})$ is a set X endowed with a collection $\mathcal{T} \subset \mathcal{P}(X)$ of subsets of X, called the *topology* of X, such that

(1) $\varnothing, X \in \mathcal{T}$,
(2) if $\mathcal{O} \subset \mathcal{T}$ then $\bigcup \mathcal{O} := \bigcup_{O \in \mathcal{O}} O \in \mathcal{T}$,
(3) if $\mathcal{O} \subset \mathcal{T}$ is finite then $\bigcap \mathcal{O} \in \mathcal{T}$,

that is, $\mathcal{T}$ contains X and $\varnothing$ and is closed under union and finite intersection. The *chaotic* or *trivial topology* is $\mathcal{T} = \mathcal{N} := \{\varnothing, X\}$, and the *discrete topology* consists of all subsets of X. If $Y \subset X$ then $\mathcal{T}_Y := \{O \cap Y \mid O \in \mathcal{T}\}$ defines the *subspace topology* or *induced topology*.

The sets $O \in \mathcal{T}$ are called *open* sets, and their complements are called *closed* sets. A set is *clopen* if it is both closed and open. If $x \in X$ then an open set containing x is said to be a *neighborhood* of x. The *closure* $\bar{A}$ of a set $A \subset X$ is the smallest closed set containing A, that is, $\bar{A} := \bigcap \{C \mid A \subset C \text{ and } C \text{ closed}\}$. x is said to be an *accumulation point* of $A \subset X$ if every neighborhood of x contains infinitely many points of A. A set is *perfect* if it is equal to the set of its accumulation points.

A *base* for the topology $\mathcal{T}$ is a subcollection $\beta \subset \mathcal{T}$ such that for every $O \in \mathcal{T}$ and $x \in O$ there exists $B \in \beta$ such that $x \in B \subset O$. A *base at* x is a subcollection $\beta_x \subset \mathcal{T}$ such that for every $O \in \mathcal{T}$ with $x \in O$ there exists $B \in \beta_x$ such that $x \in B \subset O$. $\mathcal{T}$ is said to be *first countable* if there is a countable base at x for every $x \in X$ and *second countable* if there is a countable base for $\mathcal{T}$. A topology $\mathcal{S}$ is said to be *finer* than $\mathcal{T}$ if $\mathcal{T} \subset \mathcal{S}$, *coarser* if $\mathcal{S} \subset \mathcal{T}$. If $Y \subset X$ then Y can be made into a topological space in a natural way by taking the *induced topology* $\mathcal{T}_Y := \{O \cap Y \mid O \in \mathcal{T}\}$.

A sequence $\{x_i\}_{i \in \mathbb{N}} \subset X$ is said to *converge* to $x \in X$ if for every open set O containing x there exists $N \in \mathbb{N}$ such that $\{x_i\}_{i>N} \subset O$.

$\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$ is said to be an *open cover* of X if $X = \bigcup_{\alpha \in A} O_\alpha$, and a *finite open cover* if A is finite.

Let $(X, \mathcal{T})$ be a topological space. A set $D \subset X$ is said to be *dense* in X if $\bar{D} = X$. X is said to be *separable* if it has a countable dense subset.

DEFINITION A.6.2. If $(X_\alpha, \mathcal{T}_\alpha)$, $\alpha \in A$ are topological spaces and A is any set, then the *product* or *Tychonoff topology* on $\prod_{\alpha \in A} X_\alpha$ is the topology generated by the base $\{\prod_\alpha O_\alpha \mid O_\alpha \in \mathcal{T}_\alpha,\ O_\alpha \neq X_\alpha$ for only finitely many $\alpha\}$. With or without a topology in mind we write $X^A := \prod_{\alpha \in A} X$. This is the collection of maps from

A to X. A special case is $2^A := \{0,1\}^A$, which (viewing the maps as characteristic functions) represents the collection of all subsets of A.

$\mathbb{R}^n$ with the usual open and closed sets is a familiar example of a topological space and of a product space. The open balls (or open balls with rational radius, open balls with rational center and radius) form a base. Points of a Hausdorff space are closed sets.

2. *Separation axioms, compactness.*

DEFINITION A.6.3 (Separation Axioms).

(1) $(X,\mathcal{T})$ is called a *Tychonoff space* if for any two $x_1, x_2 \in X$ there exists $O \in \mathcal{T}$ such that $x_1 \in O$ and $x_2 \notin O$.
(2) $(X,\mathcal{T})$ is called a *Hausdorff space* if for any two $x_1, x_2 \in X$ there exist $O_1, O_2 \in \mathcal{T}$ such that $x_i \in O_i$ and $O_1 \cap O_2 = \varnothing$.
(3) $(X,\mathcal{T})$ is said to be *regular* if it is Hausdorff and for any $x \in X$ and closed $C \subset X$ there exist $O_1, O_2 \in \mathcal{T}$ such that $x \in O_1$, $C \subset O_2$ and $O_1 \cap O_2 = \varnothing$.
(4) $(X,\mathcal{T})$ is said to be *normal* if it is Hausdorff and for any two closed $C_1, C_2 \subset X$ there exist $O_1, O_2 \in \mathcal{T}$ such that $C_i \subset O_i$ and $O_1 \cap O_2 = \varnothing$.

DEFINITION A.6.4 (Compactness). $(X,\mathcal{T})$ is said to be *compact* if every open cover has a finite subcover, *locally compact* if every point has a neighborhood with compact closure, and *sequentially compact* if every sequence has a convergent subsequence. X is said to be σ-compact if it is a countable union of compact sets.

PROPOSITION A.6.5. *A closed subset of a compact set is compact.*

PROOF. If K is compact, $C \subset K$ is closed, and Γ is an open cover for C then $\Gamma \cup \{K \smallsetminus C\}$ is an open cover for K, hence has a finite subcover $\Gamma' \cup \{K \smallsetminus C\}$, so Γ' is a finite subcover (of Γ) for C. □

PROPOSITION A.6.6. *A compact subset of a Hausdorff space is closed.*

PROOF. If X is Hausdorff and $C \subset X$ compact fix $x \in X \smallsetminus C$ and for each $y \in C$ take neighborhoods U_y of y and V_y of x such that $U_y \cap V_y = \varnothing$. The cover $\bigcup_{y \in C} U_y \supset C$ has a finite subcover $\{U_{x_i} \mid 0 \le i \le n\}$ and hence $N_x := \bigcap_{i=0}^n V_{y_i}$ is a neighborhood of x disjoint from C. Thus $X \smallsetminus C = \bigcup_{x \in X \smallsetminus C} N_x$ is open and C is closed. □

PROPOSITION A.6.7. *A compact Hausdorff space is normal.*

PROOF. First we show that a closed set K and a point $p \notin K$ can be separated by open sets. For $x \in K$ there are open sets O_x, U_x such that $x \in O_x$, $p \in U_x$ and $O_x \cap U_x = \varnothing$. Since K is compact there is a finite subcover $O := \bigcup_{i=1}^n O_{x_i} \supset K$, and $U := \bigcap_{i=1}^n U_{x_i}$ is an open set containing p disjoint from O. Now suppose K, L are closed sets and for $p \in L$ consider open disjoint sets $O_p \supset K$, $U_p \ni p$. By compactness of L there is a finite subcover $U := \bigcup_{j=1}^m U_{p_j} \supset L$ and $O := \bigcap_{j=1}^m O_{p_j} \supset K$ is an open set disjoint from U. □

A useful consequence of normality is the following extension result:

THEOREM A.6.8. *If X is a normal topological space, $Y \subset X$ closed, and $f\colon Y \to \mathbb{R}$ continuous (Definition A.6.15), then there is a continuous extension of f to X.*

A collection of sets is said to have the *finite intersection property* if every finite subcollection has nonempty intersection.

PROPOSITION A.6.9. *A collection of compact sets with the finite intersection property has nonempty intersection.*

PROOF. It suffices to show that in a compact space every collection of closed sets with the finite intersection property has nonempty intersection. To that end consider a collection of closed sets with empty intersection. Their complements form an open cover. Since it has a finite subcover the finite intersection property does not hold. □

DEFINITION A.6.10. The *one-point compactification* of a noncompact Hausdorff space $(X,\mathcal{T})$ is $\hat{X} := (X \cup \{\infty\}, \mathcal{S})$, where $\mathcal{S} := \mathcal{T} \cup \{(X \cup \{\infty\}) \smallsetminus K \mid K \subset X \text{ compact}\}$ and ∞ is a symbol representing an adjoined point.

It is easy to see that $\hat{X}$ is a compact Hausdorff space.

THEOREM A.6.11 (Tychonoff Theorem). *The product of compact spaces is compact.*

This result is useful in situations where a natural topology can be viewed as a product topology or is induced by a product topology. The topology of pointwise convergence is an example: the topology of pointwise convergence of maps $X \to Y$ is the Tychonoff topology of Y^X.❽

❽ Remark here on difference between cluster points of a sequence and limit of a subsequence. Relevant for distal.

While most topological spaces we consider are Hausdorff spaces, in algebraic geometry a natural topology has the Tychonoff property but is not Hausdorff.

DEFINITION A.6.12. The *Zariski topology* on $\mathbb{C}^n$ is defined by saying that $V \subset C^n$ is Zariski-closed if there are polynomials $P_1, \dots, P_m$ on C^n such that $V = \bigcap_{i=1}^m P_i^{-1}(\{0\})$. A class of sets larger than that of Zariski-closed sets is given by sets of the form $\bigcup_{i=1}^k V_i \smallsetminus V_i'$, where V_i, V_i' are Zariski-closed and $V_i' \subset V_i$ for all i; these are said to be *constructible*.

THEOREM A.6.13. *Let $\pi\colon \mathbb{C}^n \times C^m \to \mathbb{C}^n$ be the natural projection. If $V \subset \mathbb{C}^n \times C^m$ is constructible then so is $\pi(V)$.*

We digress to note in this context an analog of the Sard Theorem A.11.24.

THEOREM A.6.14. *If $\pi\colon \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$ is the natural projection and $A \subset \mathbb{C}^n \times \mathbb{C}^m$ is constructible then there exist Zariski-open $X \subset A$ and $Y \subset \mathbb{C}^n$ such that $\pi \restriction X\colon X \to Y$ is locally invertible.*

3. *Continuity.*

DEFINITION A.6.15 (Continuity). Let $(X,\mathcal{T})$ and $(Y,\mathcal{S})$ be topological spaces. A map $f\colon X \to Y$ is said to be *continuous* if $O \in \mathcal{S}$ implies $f^{-1}(O) \in \mathcal{T}$, *open* if

$O \in \mathscr{T}$ implies $f(O) \in \mathscr{S}$, *closed* if $X \smallsetminus O \in \mathscr{T}$ implies $Y \smallsetminus f(O) \in \mathscr{S}$, and a *homeomorphism* if it is continuous and bijective with continuous inverse. If there is a homeomorphism $X \to Y$ then X and Y are said to be *homeomorphic*. We denote by $C^0(X,Y)$ the space of continuous maps from X to Y and write $C^0(X)$ for $C^0(X,\mathbb{R})$. A map f from a topological space to $\mathbb{R}$ is said to be *upper semicontinuous* if $f^{-1}(-\infty, c) \in \mathscr{T}$ for all $c \in \mathbb{R}$, *lower semicontinuous* if $f^{-1}(c,\infty) \in \mathscr{T}$ for $c \in \mathbb{R}$.

A property of a topological space that is the same for any two homeomorphic spaces is said to be a *topological invariant*.

PROPOSITION A.6.16. *The image of a compact set under a continuous map is compact.*

PROOF. If C is compact and $f\colon C \to Y$ continuous and surjective then any open cover Γ of Y induces an open cover $f_*\Gamma := \{f^{-1}(O) \mid O \in \Gamma\}$ of C which by compactness has a finite subcover $\{f^{-1}(O_i) \mid i = 1,\dots,n\}$. By surjectivity $\{O_i\}_{i=1}^n$ is a cover for Y. □

A useful application of the notions of continuity, compactness, and being Hausdorff is the following result:

PROPOSITION A.6.17 (Invariance of domain). *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

PROOF. Suppose X is compact, Y Hausdorff, $f\colon X \to Y$ bijective and continuous, and $O \subset X$ open. Then $C := X \smallsetminus O$ is closed, hence compact, and $f(C)$ is compact, hence closed, so $f(O) = Y \smallsetminus f(C)$ (by bijectivity) is open. □

4. *Connectedness.*

DEFINITION A.6.18 (Connectedness). A topological space $(X,\mathscr{T})$ is said to be *connected* if no two disjoint open sets cover X. (Equivalently, there is no proper clopen subset.)

$(X,\mathscr{T})$ is said to be *path-connected* if for any two points $x_0, x_1 \in X$ there exists a continuous curve $c\colon [0,1] \to X$ with $c(i) = x_i$. A *connected component* is a maximal connected subset of X.

$(X,\mathscr{T})$ is said to be *totally disconnected* if every point is a connected component. A perfect, totally disconnected compact space it called a *Cantor set*.

The ternary Cantor set and $\mathbb{Q} \subset \mathbb{R}$ are totally disconnected. It is not hard to see that connected components are closed. Thus connected components are open if there are only finitely many and, more generally, if every point has a connected neighborhood (that is, the space is locally connected). This is not the case for $\mathbb{Q}$.

THEOREM A.6.19. *Every Cantor set is homeomorphic to the ternary Cantor set.*

THEOREM A.6.20. *A continuous image of a connected space is connected.*

REMARK A.6.21. This implies that a path-connected space is connected. The converse is false as is shown by the closure of the graph of $\sin 1/x$ in $\mathbb{R}^2$.

THEOREM A.6.22. *The product of two connected topological spaces is connected.*

5. *Manifolds.*

DEFINITION A.6.23. A *topological manifold* is a Hausdorff space X with a countable base for the topology such that every point is contained in an open set homeomorphic to a ball in $\mathbb{R}^n$. A pair (U, h) of such a neighborhood and a homeomorphism $h\colon U \to B \subset \mathbb{R}^n$ is called a *chart* or a system of *local coordinates*. A topological *manifold with boundary* is a Hausdorff space X with a countable base for the topology such that every point is contained in an open set homeomorphic to an open set in $\mathbb{R}^{n-1} \times [0,\infty)$.

REMARK A.6.24. One easily sees that if X is connected then n is constant. In this case it is called the *dimension* of the topological manifold. Path connectedness and connectedness are equivalent for topological manifolds.

DEFINITION A.6.25. Consider a topological space $(X,\mathcal{T})$ and suppose there is an equivalence relation $\sim$ defined on X. Then there is a natural projection π to the set $\hat{X}$ of equivalence classes. The *identification space* or *factor space* $X/\sim := (\hat{X},\mathcal{S})$ is the topological space obtained by saying that a set $O \subset \hat{X}$ is open if $\pi^{-1}(O)$ is open, that is, taking on $\hat{X}$ the finest topology with which π is continuous.

An important class of factor spaces appears when there is a group G of homeomorphisms acting on X such that the orbits are closed. Then one identifies points on the same orbit and obtains an identification space which in this case is denoted by X/G and called the *quotient* of X by G. For the case where $X = S^1$ and G is the cyclic group of iterates of a rational rotation we get $X/G \simeq X$. If $X = \mathbb{R}^2$ and G is the group of translations parallel to the x-axis then $X/G \simeq \mathbb{R}$. The torus is obtained from $\mathbb{R}^n$ by identifying points modulo $\mathbb{Z}^n$, that is, two points are equivalent if their difference is in $\mathbb{Z}^n$. Equivalently, one obtains it from identifying pairs of opposite sides of the unit square (or any rectangle) with the same orientation. A *cone* over a space X is the space obtained from identifying all points of the form $(x,1)$ in $(X \times [0,1]$, product topology). The sphere is obtained by identifying all boundary points of a closed ball.

A priori the topology of an identification space may not be very nice. Unless the equivalence classes are all closed, for example, the identification space is not even a Hausdorff space. In particular the space of orbits of a dynamical system with some recurrent behavior is not a very good object from the topological point of view. An example is X/G, where $X = S^1$ and G is the group of iterates of an irrational rotation.

b. Homotopy theory.

DEFINITION A.6.26. Two continuous maps $h_0, h_1 \colon X \to Y$ between topological spaces are said to be *homotopic* if there exists a continuous map $h \colon [0,1] \times X \to Y$ (the *homotopy*) such that $h(i,\cdot) = h_i$ for $i = 0,1$. If $h_0(x) = h_1(x) = p$ for some $x \in X$ then $h_0, h_1 \colon X \to Y$ are said to be *homotopic rel p* if h can be chosen such that $h(\cdot, x) = p$. If $X = [0,1]$, $h_0(0) = h_1(0)$, and $h_0(1) = h_1(1)$ then we say that h_0, h_1 are *homotopic rel endpoints* if $h(\cdot,0)$ and $h(\cdot,1)$ can be taken constant. h is said to be *null-homotopic* if it is homotopic to a constant map. If h_1, h_2 are homeomorphisms they are said to be *isotopic* if h can be taken such that every $h(t,\cdot)$ is a homeomorphism. X and Y are said to be *homotopically equivalent* if there exist maps $g \colon X \to Y$ and $h \colon Y \to X$ such that $g \circ h$ and $h \circ g$ are homotopic to the identity. X is said to be *contractible* if it is homotopic to a point. A property of a topological space which is the same for any homotopically equivalent space is called a *homotopy invariant*.

Obviously homeomorphic spaces are homotopically equivalent. The circle and the cylinder are homotopically equivalent but not homeomorphic. Balls and cones are contractible. Contractible spaces are connected.

DEFINITION A.6.27. Let M be a topological manifold, $p \in M$, and consider the collection of curves $c \colon [0,1] \to M$ with $c(0) = c(1) = p$. If c_1 and c_2 are such curves then let $c_1 \cdot c_2$ be the curve given by

$$c_1 \cdot c_2(t) := \begin{cases} c_1(2t) & \text{when } t \le \frac{1}{2}, \\ c_2(2t-1) & \text{when } t \ge \frac{1}{2}. \end{cases}$$

Upon identifying curves homotopic rel endpoints one obtains a group called the *fundamental group* $\pi_1(M,p)$ of M at p. A space with trivial fundamental group is said to be *simply connected*, 1-connected if it is also connected.

We are mostly interested in connected manifolds where path-connectedness ensures that the groups obtained at different p are isomorphic. Thus we simply write $\pi_1(M)$. Since the fundamental group is defined modulo homotopy, it is the same for homotopically equivalent spaces, that is, it is a homotopy invariant. The *free* homotopy classes of curves (that is, with no fixed base point) correspond exactly to the conjugacy classes of curves modulo changing base point, so there is a natural bijection between the classes of freely homotopic closed curves and conjugacy classes in the fundamental group.

DEFINITION A.6.28. If M, M' are topological manifolds and $\pi \colon M' \to M$ is a continuous map such that $\operatorname{card} \pi^{-1}(y)$ is independent of $y \in M$ and every $x \in \pi^{-1}(y)$ has a neighborhood on which π is a homeomorphism to a neighborhood of $y \in M$ then M' (or (M', π)) is said to be a *covering (space)* or *cover* of M. If $n = \operatorname{card} \pi^{-1}(y)$ is finite then (M', π) is said to be an *n-fold covering*. If $f \colon N \to M$ is continuous and $F \colon N \to M'$ is such that $f = \pi \circ F$ then F is said to be a *lift* of f. If $f \colon M \to M$ is continuous and $F \colon M' \to M'$ is continuous such that $f \circ \pi = \pi \circ F$ then F is said to be a *lift* of f as well. A simply connected covering is called the *universal cover*. A homeomorphism of a covering M' of M is called a *deck transformation* if it is a lift of the identity on M.

REMARK A.6.29 (Examples). $(\mathbb{R}, \exp(2\pi i(\cdot)))$ is a covering of the unit circle. Geometrically one can view this as the helix $(e^{2\pi i x}, x)$ covering the unit circle under projection. The map defined by taking the fractional part likewise defines a covering of the circle $\mathbb{R}/\mathbb{Z}$ by $\mathbb{R}$. The torus is covered by the cylinder which is in turn covered by $\mathbb{R}^2$. Notice that the fundamental group $\mathbb{Z}$ of the cylinder is a subgroup of that of the torus ($\mathbb{Z}^2$) and $\mathbb{R}^2$ is a simply connected cover of both. The expanding maps on the circle (2.8.1) define coverings of the circle by itself. Factors of the Poincaré upper half-plane are covered by the upper half-plane (Subsection 7.3e).

There is a natural bijection between conjugacy classes of subgroups of $\pi_1(M)$ and classes of covering spaces modulo homeomorphisms commuting with deck transformations. In particular the universal cover is unique. This bijection can be described as follows. Suppose (M', π) is a covering of M and $x_0, x_1 \in \pi^{-1}(y)$. Since M' is path-connected there are curves $c\colon [0,1] \to M'$ with $c(i) = x_i$ for $i = 1, 2$. Under π these project to loops on M. Any continuous map induces a homomorphism between the fundamental groups. Any continuous map possesses a lift, so a homotopy of the loop $\pi \circ c$ rel y can be lifted to a homotopy of the curve c and since by hypothesis $\pi^{-1}(y)$ is discrete, this homotopy is a homotopy rel endpoints. In particular homotopic curves project to homotopic curves and, by considering the case $x_1 = x_2$, the fundamental group of M' injects into the fundamental group of M as a subgroup. This is the subgroup corresponding to the covering. Furthermore this subgroup is a proper subgroup whenever π is not a homeomorphism, that is, the cover is a nontrivial covering. Thus a simply connected space has no proper coverings. One can also see that any two coverings M_1' and M_2' of M have a common covering M'', so the universal cover is unique. Any topological manifold has a universal cover.

c. Metric spaces. For several quite natural notions a topological structure is not adequate, but one rather needs a *uniform* structure, that is, a topology in which one can compare neighborhoods of different points. This can be defined abstractly and is realized for topological vector spaces (see Definition A.10.1), but it is a little more convenient to introduce these concepts for metric spaces.

1. *Metric, completeness.*

DEFINITION A.6.30 (Metric). If X is a set then $d\colon X \times X \to \mathbb{R}$ is called a *metric* if

(1) $d(x,y) = d(y,x)$,
(2) $d(x,y) = 0 \Leftrightarrow x = y$,
(3) $d(x,y) + d(y,z) \geq d(x,z)$ *(triangle inequality).*

If d is a metric then (X,d) is said to be a *metric space*. The set $B(x,r) := \{y \in X \mid d(x,y) < r\}$ is called the *(open) r-ball* around x.

$O \subset X$ is said to be *open* if for every $x \in O$ there exists $r > 0$ such that $B(x,r) \subset O$. A toplogical space is said to be *metrizable* if the topology consists of the open sets for a metric. (For example, the discrete metric $\delta(x,y) = 1$ if $x \neq y$, $\delta(x,x) = 0$ induces the discrete topology.)

For $A \subset X$ the set $\bar{A} := \{x \in X \mid \forall r > 0 \quad B(x,r) \cap A \neq \varnothing\}$ is called the *closure* of A. A is said to be *closed* if $\bar{A} = A$.

A sequence $\{x_i\}_{i \in \mathbb{N}}$ is said to be a *Cauchy sequence* if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_i, x_j) < \epsilon$ whenever $i, j \geq \mathbb{N}$. X is said to be *complete* if every Cauchy sequence converges. A complete separable metric space (or a topological space homeomorphic to such a space) is called a *Polish space*.

REMARK A.6.31. We say that x is close to y if $d(x,y) < \epsilon$, with ϵ not specified but understood to be small. For instance, to say that a property $P(x)$ holds for all x sufficiently close to y means that there is an $\epsilon > 0$ such that $P(x)$ for all $x \in B(y,\epsilon)$.

The collection of open sets induces a topology with the open balls as a base. Closed sets have open complements. The definitions are consistent with those made for topological spaces. For metric spaces the notions of compactness and sequential compactness are equivalent.

DEFINITION A.6.32. For an open cover of a compact metric space there exists a number δ such that every δ-ball is contained in an element of the cover. The largest such number is called the *Lebesgue number* of the cover.

Completeness is an important property since it allows us to perform limit operations which arise frequently in our constructions. Notice that it is not possible to define a notion of Cauchy sequences in an arbitrary topological space since one lacks the possibility of comparing neighborhoods at different points. A useful observation is that compact sets are complete by sequential compactness.

THEOREM A.6.33 (Baire Category Theorem). *In a complete metric space a countable intersection of open dense sets is dense. The same holds for a locally compact Hausdorff space.*

PROOF. If $\{O_i\}_{i \in \mathbb{N}}$ are open and dense in X and $\varnothing \neq B_0 \subset X$ is open then inductively choose a ball B_{i+1} of radius at most ϵ/i such that $\bar{B}_{i+1} \subset O_{i+1} \cap B_i$. The centers converge by completeness, so $\varnothing \neq \bigcap_i \bar{B}_i \subset B_0 \cap \bigcap_i O_i$. For locally compact Hausdorff spaces take B_i open with compact closure and use the finite intersection property. □

PROPOSITION A.6.34. *Any metric space is normal and hence Hausdorff. A metric space has a countable base if and only if it is separable.*

Conversely (using Proposition A.6.7) we have

PROPOSITION A.6.35. *A normal space with a countable base for the topology, hence any compact Hausdorff space with a countable base, is metrizable.*

2. *Continuous maps.*

DEFINITION A.6.36 (Regularity). Let (X,d), (Y,dist) be metric spaces. A map $f: X \to Y$ is said to be *uniformly continuous* if for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in X$ with $d(x,y) < \delta$ we have $\mathrm{dist}(f(x), f(y)) < \epsilon$. A uniformly continuous bijection with uniformly continuous inverse is said to be a *uniform homeomorphism*.

A family $\mathscr{F}$ of maps $X \to Y$ is said to be *equicontinuous* if for every $x \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $\operatorname{dist}(f(x), f(y)) < \epsilon$ for all $y \in X$ and $f \in \mathscr{F}$.

A map $f\colon X \to Y$ is said to be *Hölder continuous* with exponent α, or α-*Hölder*, if there exist $C, \epsilon > 0$ such that $d(x, y) < \epsilon$ implies $\operatorname{dist}(f(x), f(y)) \le C(d(x, y))^{\alpha}$, *Lipschitz continuous* if it is 1-Hölder, in which case C is said to be a *Lipschitz constant* of f, and the smallest of these is called *the* Lipschitz constant. A Hölder continuous map with Hölder continuous inverse is said to be bi-Hölder, and we say that f is *bi-Lipschitz* if it is Lipschitz and has a Lipschitz inverse.

A map $f\colon X \to Y$ is said to be an *isometry* if $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

For a map $g\colon S^1 \to X$ or $g\colon I \to X$, where $I \subset \mathbb{R}$, the *total variation* is $\operatorname{Var}(g) := \sup \sum_{k=1}^{n} d(g(x_k), g(x'_k))$. Here the sup is taken over all finite collections $\{x_k, x'_k\}_{k=1}^{n}$ such that x_k, x'_k are endpoints of an interval I_k and $I_k \cap I_j = \varnothing$ for $k \neq j$. The function is said to be of *bounded variation* if $\operatorname{Var}(g)$ is finite.

REMARK A.6.37. Every Lipschitz function and hence every continuously differentiable function has bounded variation.

It is not hard to show

PROPOSITION A.6.38. *A uniformly continuous map from a subset of a metric space uniquely extends to the closure.*

A metric space can be made complete in the following way:

DEFINITION A.6.39. If X is a metric space and there is an isometry from X onto a dense subset of a complete metric space $\hat{X}$ then $\hat{X}$ is called the *completion* of X.

Up to isometry the completion of X is unique: If there are two completions X_1 and X_2 then by construction there is a bijective isometry between dense subsets and therefore this isometry extends (by uniform continuity) to the entire space. On the other hand completions always exist by virtue of the construction used to obtain the real numbers from the rational numbers. This completion is obtained from the space of Cauchy sequences on X by identifying two sequences if the distance between corresponding elements converges to zero. The distance between two (equivalence classes of) sequences is defined as the limit of the distance between corresponding elements. The isometry maps points to constant sequences.

DEFINITION A.6.40. If X is a compact metrizable topological space (for example, a compact manifold), then the space $C(X, Y)$ of continuous maps from X to a metric space Y possesses the C^0 or *uniform* topology. It arises by fixing a metric ρ in Y and defining the distance d between $f, g \in C(X, Y)$ by

$$d(f, g) := \max_{x \in X} \rho(f(x), g(x)).$$

We define $C(X) := C(X, \mathbb{R})$.

REMARK A.6.41. The subset $\mathrm{Hom}(X)$ of $C(X, X)$ of homeomorphisms of X is neither open nor closed in the C^0 topology. It possesses, however, a natural topology as a complete metric space induced by the metric

$$d_H(f, g) := \max(d(f, g), d(f^{-1}, g^{-1})).$$

THEOREM A.6.42. *If X is a compact Hausdorff space then the following are equivalent.*

(1) *X is metrizable.*
(2) *X is second countable.*
(3) *$C(X)$ and $C(X, \mathscr{C})$ are separable.*

If X is σ-compact we introduce the compact–open topologies for maps and homeomorphisms, that is, the topologies of uniform convergence on compact sets.

We sometimes use the fact that equicontinuity and uniform boundedness give some compactness of a family of continuous functions in the uniform topology.

THEOREM A.6.43 (Arzelá–Ascoli Theorem). *Let X, Y be metric spaces, X separable, and $\mathscr{F}$ an equicontinuous family of maps. If $\{f_i\}_{i\in\mathbb{N}} \subset \mathscr{F}$ such that $\{f_i(x)\}_{i\in\mathbb{N}}$ has compact closure for every $x \in X$ then there is a subsequence converging uniformly on compact sets to a function f.*

Thus in particular a closed bounded equicontinuous family of maps on a compact space is compact in the uniform topology (induced by the maximum norm).

SKETCH OF PROOF. Since $\{f_i(x)\}_{i\in\mathbb{N}}$ has compact closure for every point x of a countable dense subset S of X, a diagonal argument shows that there is a subsequence f_{i_k} which converges at every point of S. Use equicontinuity to show that for every $x \in X$ the sequence $f_{i_k}(x)$ is Cauchy, hence convergent (since $\{f_i(x)\}_{i\in\mathbb{N}}$ has compact, hence complete, closure). Using equicontinuity again yields continuity of the pointwise limit. Finally a pointwise convergent equicontinuous sequence converges uniformly on compact sets. □

3. *Hausdorff metric.* We introduce the following metric in the space of closed subsets of a metric space X:

DEFINITION A.6.44. The *Hausdorff metric* is defined by setting❾

$$d(A, B) := \max\{\sup\{d(x, B) \mid x \in A\}, \sup\{d(A, y) \mid y \in B\}\}$$

for any two closed sets A, B. We refer to a limit with respect to the topology induced by the Hausdorff metric as a *Hausdorff limit.*

❾ We replaced "+" by "max"; check whether this necessitates changes elsewhere.

We make two observations:

LEMMA A.6.45. *The Hausdorff metric on the closed subsets of a compact metric space defines a compact topology.*

PROOF. We need to verify total boundedness and completeness. Pick a finite $\epsilon/2$-net N. Any closed set $A \subset X$ is covered by a union of ϵ-balls centered at points of N and the closure of the union of these has Hausdorff distance at most ϵ from A. Since there are only finitely many such sets, we have shown that this metric is totally bounded. To show that it is complete consider a Cauchy sequence (with respect to the Hausdorff metric) of closed sets $A_n \subset X$. If we let $A := \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{n\geq k} A_n}$ then one can easily check that $d(A_n, A) \to 0$. □

Notice that any homeomorphism f of a compact metric space X induces a natural homeomorphism of collection of closed subsets of X with the Hausdorff metric, so we may conclude the following:

LEMMA A.6.46. *The set of closed invariant sets of a homeomorphism f of a metric space is a closed set with respect to the Hausdorff metric.*

PROOF. This is just the set of fixed points of the induced homeomorphism, hence is closed. □

7. Cantor sets and sequence spaces

a. Sequence spaces. For each natural number $N \geq 2$ consider the space

$$\Omega_N := \{0,\dots,N-1\}^{\mathbb{Z}} = \{\omega = (\dots,\omega_{-1}.\omega_0,\omega_1,\dots) \mid \omega_i \in \{0,\dots,N-1\} \text{ for } i \in \mathbb{Z}\}$$

of two-sided sequences of N symbols and a similar one-sided space

$$\Omega_N^R := \{0,\dots,N-1\}^{\mathbb{N}_0} = \{\omega = (\omega_0,\omega_1,\omega_2,\dots) \mid \omega_i \in \{0,\dots,N-1\} \text{ for } i \in \mathbb{N}_0\}.$$

The standard topology on these spaces is obtained as the product topology arising from the discrete topology on $\{0,1,\dots,N-1\}$. Thus, Ω_N is a Cantor set (Definition A.6.18).

Since $\{0,1,\dots,N-1\} = \mathbb{Z}/n\mathbb{Z}$ is a group, these sequence spaces, being products, are compact abelian topological groups.

Fix integers $n_1 < n_2 < \cdots < n_k$ and numbers $\alpha_1,\dots,\alpha_k \in \{0,1,\dots,N-1\}$ and call the subset

$$C^{n_1,\dots,n_k}_{\alpha_1,\dots,\alpha_k} := \{\omega \in \Omega_N \mid \omega_{n_i} = \alpha_i \text{ for } i = 1,\dots,k\} \tag{A.7.1}$$

a *cylinder* and the number k of fixed digits the *rank* of that cylinder. Cylinders in the space Ω_N^R are defined similarly. Then cylinders are open sets and form a base for the topology. Thus, every cylinder is also closed because the complement to a cylinder is a finite union of cylinders. The most general open set is a countable union of cylinders.

The topology is metrizable, for instance by the product metric

$$d_\lambda(\omega,\omega') = \sum_{n=-\infty}^{\infty} \frac{|\omega_n - \omega'_n|}{\lambda^{|n|}}$$

for any fixed $\lambda > 1$. Indeed, we have

PROPOSITION A.7.1 (Cylinders are balls). *If $\lambda > 2N-1$, then $C_{\alpha_{1-n}\dots\alpha_{n-1}} = B_{d_\lambda}(\alpha,\lambda^{1-n})$, the λ^{1-n}-ball around α.*

PROOF. If $\omega \in C_{\alpha_{1-n}\dots\alpha_{n-1}}$ then

$$d_\lambda(\alpha,\omega) = \sum_{i\in\mathbb{Z}} \frac{|\alpha_i,\omega_i|}{\lambda^{|i|}} = \sum_{|i|\geq n} \frac{|\alpha_i,\omega_i|}{\lambda^{|i|}} \leq \sum_{|i|\geq n} \frac{N-1}{\lambda^{|i|}} = \frac{1}{\lambda^{n-1}}\frac{2(N-1)}{\lambda-1} < \frac{1}{\lambda^{n-1}}.$$

Thus $C_{\alpha_{1-n}\dots\alpha_{n-1}} \subset B_{d_\lambda}(\alpha,\lambda^{1-n})$. If $\omega \notin C_{\alpha_{1-n}\dots\alpha_{n-1}}$ then

$$d_\lambda(\alpha,\omega) = \sum_{i\in\mathbb{Z}} \frac{\delta(\alpha_i,\omega_i)}{\lambda^{|i|}} \geq \lambda^{1-n}$$

because $\omega_i \neq \alpha_i$ for some $|i| < n$. Thus $\omega \notin B_{d_\lambda}(\alpha,\lambda^{1-n})$. □

The corresponding result (with the same proof) also holds for the product metric obtained from the discrete metric $\delta(k,l) = 1$ if $k \neq l$, $\delta(k,k) = 0$ on $\{0,\dots,N-1\}$.

PROPOSITION A.7.2 (Cylinders are balls). *If*

$$d_\lambda(\omega,\omega') := \sum_{i\in\mathbb{Z}} \frac{\delta(\omega_i,\omega'_i)}{\lambda^{|i|}},$$

for some $\lambda > 3$, *then* $C_{\alpha_{1-n}\dots\alpha_{n-1}} = B_{d_\lambda}(\alpha,\lambda^{1-n})$.

The choice of λ in these metrics is not particularly important:

PROPOSITION A.7.3 (Hölder structure). *For any* λ,μ *the identity map* $\mathrm{Id}\colon (\Omega_N,d_\lambda) \to (\Omega_N,d_\mu)$ *is Hölder continuous, that is, there exist* $a,c>0$ *such that for any* $\omega,\omega' \in \Omega_N$ *we have*

$$d_\mu(\omega,\omega') < c\, d_\lambda(\omega,\omega')^a. \tag{A.7.2}$$

The different metrics d_λ not only define the same topology on Ω_N (although they are not equivalent as metrics) but also determine a Hölder structure. This means that the notion of Hölder-continuous function (Definition A.6.36) with respect to the metric d_λ does not depend on λ. That class of Hölder-continuous functions plays an extremely important role in applications to differentiable dynamics (see 23 and Chapter 24) and can be described as follows.

DEFINITION A.7.4. Let $K \subset \Omega_N$ be closed and $\varphi \in C(K,\mathbb{C})$. For $n = 0,1,\dots$ let

$$V_n(\varphi) := \max\{|\varphi(\omega)-\varphi(\omega')| \mid \omega_k = \omega'_k \text{ for } |k| \leq n\},$$

where $\omega = (\dots,\omega_{-1}.\omega_0,\omega_1,\dots)$, $\omega' = (\dots,\omega'_{-1}.\omega'_0,\omega'_1,\dots) \in K$. Since Ω_N is compact, φ is uniformly continuous and $V_n(\varphi) \to 0$ as $n \to \infty$. We say that φ has *exponential type* if for some $a,c > 0$

$$V_n(\varphi) \leq ce^{-an}.$$

It is not difficult to see that φ has exponential type if and only if it is Hölder continuous (Definition A.6.36) with respect to some (and hence any) metric d_λ (see **??**).

All of the above discussion translates with obvious changes to the spaces Ω_N^R.

b. Linear Cantor sets. The *topology* of Cantor sets has been discussed in Subsection A.6a. We now discuss the *geometry* of Cantor sets in $\mathbb{R}$ (with the ambient metric). The motivation is that performing the Cantor construction while taking away the middle-$1/n!$ piece produces a Cantor set that is "thicker" than the same construction with middle-$1-\frac{1}{n!}$ pieces removed.

DEFINITION A.7.5 (Thickness). Two intervals are said to be *linked* if the interior of each contains exactly one endpoint of the other. Two Cantor sets are said to be linked if their convex hulls are linked.

If $C \subset \mathbb{R}$ is a Cantor set then each connected component of $\mathbb{R} \smallsetminus C$ is called a *complementary interval*.

Two Cantor sets in $\mathbb{R}$ are said to *overlap* if neither of them lies in a complementary interval of the other.

Each bounded complementary interval is called a *gap*.

If $x \in C$ is an endpoint of a gap I then the *bridge* $B(x)$ of C at x is the maximal interval (in $\mathbb{R}$) with x as an endpoint and disjoint from all complementary intervals whose length $l(B(x))$ is that of I or more.

The *thickness* of C at an endpoint x of a gap I is then defined by

$$\tau(C,x) := \frac{l(B(x))}{l(I)},$$

and the *thickness* of C is defined as $\tau(C) := \inf_x \tau(C,x)$.

REMARK A.7.6. Note that linked Cantor sets overlap and that being linked is an open condition (using the Hausdorff metric). Also, the thickness of a Cantor set is invariant under linear maps.

REMARK A.7.7. For studying the behavior of thickness under nonlinear maps an alternative definition is useful because even if lengths of gaps are changed by only slightly different ratios under the map, bridges might not be mapped to bridges because of a change in the order of lengths of gaps. Instead, one can fix a labeling: We say that an ordering $\mathscr{I} := (I_n)_{n\in\mathbb{N}}$ of the gaps is a *presentation* of C, and for an endpoint $x \in C$ of a gap I_n the *x-component* $C(x)$ of C at x is the maximal interval (in $\mathbb{R}$) with x as an endpoint and disjoint from all gaps I_j for $j < n$. Then $\tau(C,\mathscr{I},x) := l(C(x))/l(I)$ and

$$\tau(C) = \sup\{\inf\{\tau(C,\mathscr{I},x) \mid x \text{ is a gap endpoint}\} \mid \mathscr{I} \text{ is a presentation}\}$$

because the supremum is attained for any presentation that is monotone in gap lengths.

The prime application of the notion of thickness is that two sufficiently thick overlapping Cantor sets must intersect.

PROPOSITION A.7.8 (Newhouse Gap Lemma). *If $C_1, C_2 \subset \mathbb{R}$ are overlapping Cantor sets with $\tau(C_1)\tau(C_2) > 1$ then $C_1 \cap C_2 \neq \varnothing$.*

PROOF. Consider the gaps of C_1. If none of them contains a point of C_2 then $C_2 \subset C_1$ and we are done. Otherwise, consider a gap I_1 of C_1 that contains a point of C_2 and hence an endpoint of a gap of C_2. Of the gaps of C_2 with an endpoint in

I_1, there is at least one that is not contained in I_1, for otherwise $C_2 \subset I_1$ contrary to the overlap assumption. Calling this gap I_2 we have what is called a *gap pair* (I_1, I_2): I_1 and I_2 are linked.

CLAIM A.7.9. *If (I_1, I_2) is a gap pair then there exists one of the following:*

(1) *a point in $C_1 \cap C_2$,*
(2) *a gap pair (I_1', I_2) with $l(I_1') < l(I_1)$,*
(3) *a gap pair (I_1, I_2') with $l(I_2') < l(I_2)$.*

PROOF. If $x_i \in I_{3-i}$ is an endpoint of I_i for $i = 1, 2$ and $B_i(\cdot)$ denotes bridges in C_i, then the thickness assumption implies

$$\frac{l(B_1(x_1))}{l(I_2)} \cdot \frac{l(B_2(x_2))}{l(I_1)} = \frac{l(B_1(x_1))}{l(I_1)} \cdot \frac{l(B_2(x_2))}{l(I_2)} > 1.$$

Thus, at least one of the fractions on the left exceeds 1, and to fix ideas we consider the case of $l(B_1(x_1))/l(I_2) > 1$. If $y \neq x_2$ is the other endpoint of I_2 this implies that $y \in B_1(x_1)$. There are two possibilities. Either $y \in C_1$ and we are in the first case (since $y \in C_2$) or y is in a gap I_1' of C_1 of length less than $l(I_1)$. This gives the desired gap pair (I_1', I_2) with $l(I_1') < l(I_1)$. (The third possibility in the claim arises from the other fraction exceeding 1.) □

To prove Proposition A.7.8, start with the original gap pair and apply the claim. If it yields a point in $C_1 \cap C_2$ we are done. Otherwise, we can apply the claim to the new gap pair. Either this recursion ends in case 1 after finitely many steps, and the proof is complete, or it gives a sequence of gap pairs. Since the gap lengths are summable we get either a sequence $I_1^{(i)}$ with $l(I_1^{(i)}) \to 0$ or a sequence $I_2^{(i)}$ with $l(I_2^{(i)}) \to 0$. Either way, the midpoints of these intervals necessarily converge to a point of $C_1 \cap C_2$, which completes the proof in this last case. □

8. The Perron–Frobenius theorem for positive matrices

THEOREM A.8.1 (Perron–Frobenius Theorem). *Let L be an $N \times N$ matrix with nonnegative entries such that for some power L^{n_0} all entries are positive. Then L has one (up to a scalar) eigenvector e with positive coordinates and no other eigenvectors with nonnegative coordinates. Moreover, the eigenvalue corresponding to e is simple, positive, and greater than the absolute values of all other eigenvalues.*

PROOF. Denote by P the set of all vectors in $\mathbb{R}^N$ with nonnegative coordinates and by σ the unit simplex in P, that is,

$$\sigma := \{(x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_i \geq 0, \ \|x\|_1 := \sum_{i=1}^{N} |x_i| = 1\}.$$

By assumption $LP \subset P$. Thus for every $x \in \sigma$ there exists a unique vector $Tx \in \sigma$ proportional to Lx, and we have a canonical map $T\colon \sigma \to \sigma$. Since $L^{n_0}(P) \subset \operatorname{Int} P$ we have $T^{n_0}(\sigma) \subset \operatorname{Int} \sigma$.

If $S \subset \sigma$ is convex, then so are TS and $T^{-1}(S)$, and $\operatorname{ex}(T(S)) \subset T(\operatorname{ex}(S))$ (see Definition A.10.25).