MASS-11; ANALYSIS

FALL 2011

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HOMEWORK # 1

Due on Wednesday August 31

1. Consider two norms in the standard n-dimensional vector space $\mathbb{R}^n$: Euclidean norm $\|(x_, \dots, x_n)\|_2 = (\sum_{i=1}^n x_i^2)^{1/2}$ and maximal norm $\|(x_, \dots, x_n)\|_\infty = \max(|x_1, \dots, |x_n|)$.

Prove that there exist a constant $C_n > 0$ such that

$$C_n\|(x_, \dots, x_n)\|_2 \le \|(x_, \dots, x_n)\|_\infty \le C_n^{-1}\|(x_, \dots, x_n)\|_2.$$

2. Let p be a positive number. For $x = (x_1, x_2) \in \mathbb{R}^2$ define

$$\|x\|_p = (|x_1|^p + |x_2|^p)^{1/p}.$$

For what values of p $\|x\|_p$ is a norm? Give the answer with complete proof.

3. Prove that the space of polynomials of all degrees on the interval [0,1] with the metric $d(p, q) = \max_{0\le x\le 1} |p(x) - q(x)|$ is not complete.

4. Prove that the closed unit ball space of continuous functions on the unit interval $C([0, 1])$ with the metric $d(f, g) = \max_{0\le x\le 1} |f(x) - g(x)|$ (i.e all functions such that $\max |f(x)| \le 1$) is not compact.

5. Recall that a metric space is *separable* if it contains an everywhere dense countable subset.

(1) Prove that the space $C([0, 1])$ is separable.
(2) Prove that the space of bounded continuous functions on the real line $C(\mathbb{R})$ with the metric $d(f, g) = \sup_{x\in\mathbb{R}} |f(x) - g(x)|$ is not separable.

NOTE CORRECTIONS TO THE PRINTED VERSION!

EXTRA CREDIT PROBLEM (Due September 7)

1*. Find a proper generalization of problems **4** and **5** to spaces of functions on general metric spaces.