MASS-11; ANALYSIS

FALL 2011

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HOMEWORK # 10

Due on Wednesday November 16

Let $\alpha > 0$. A function f on a metric space (X, d) is called α-*Hoelder* if there exists a constant C such that for every $x, y \in X$

$$|f(x) - f(y)| \leq C(d(x, y))^{\alpha}.$$

More generally, given a modulus of continuity ρ, a function is ρ-*continuous* if it has modulus of continuity $C\rho$ for some C.

44. Prove that for any modulus of continuity ρ the ρ-continuous functions are dense in the space of $C(K)$ where K is the Cantor set with any metric.

45. Prove that there exists a metric d on the unit interval such that for *any* α α-continuous functions are dense in $C([0, 1])$.

Note: remember that for the standard metric this is only true for $\alpha \leq 1$.

46. Prove the following generalization of the Weierstrass-Stone theorem:

Let $\mathcal{A}$ be an algebra of real-valued functions on a compact metric space X. Say that points x, y are equivalent if $f(x) = f(y)$ for any $f \in A$.

Then either the closure of $\mathcal{A}$ consists of all functions constant on the equivalence classes of points or there is exactly one equivalence class E where all functions form $\mathcal{A}$ vanish and the closure of $\mathcal{A}$ consists of all functions vanishing at E.

47.

(1) Prove that any closed convex subset of a Hilbert space is closed in weak topology.
(2) Give an example of a closed set in a Hilbert space that is not closed in weak topology.

48. Prove that there exists a function f on the unit interval with modulus of continuity $\sqrt{t}$ and norm one in $C([0,1])$ that cannot be represented as $\frac{g+h}{2}$ where $g \neq f$ and both g and h have the same properties as f, i.e. norm one and modulus of continuity $\sqrt{t}$.