MASS-11; ANALYSIS

FALL 2011

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HOMEWORK # 2

Due on Wednesday September 6

6. Let p be a prime number and let $a_1 \in \{0, \dots, p-1\}$, $i = 0, 1 \dots$.

(1) Prove that the series

$$\sum_{n=0}^{\infty} a_n p^n$$

converges in the space $\mathbb{Q}_p$ of p-adic numbers.

(2) Prove that for every p-adic number x there exists an integer K such that x has unique representation of the form

$$\sum_{n=K}^{\infty} a_n p^n$$

with $a_n \in \{0, \dots, p-1\}$ with $a_K \neq 0$.

7. Prove that every triangle in $\mathbb{Q}_p$ is isosceles, i.e at least two of its sides have equal lengths.

8. Let m be an arbitrary integer. Define m-adic norm $\|\cdot\|_m$ on $\mathbb{Q}$ as follows: represent $\frac{a}{b}$, with a and b are relatively prime, as $m^n \frac{a'}{b'}$ where b' is relatively prime both with a' and with m. Then $\|\frac{a}{b}\|_m = m^{-n}$.

Prove that $\|\cdot\|_m$ is a norm and show that if m is not prime the norm of the product is not necessarily equal to the product of norms.

9. Consider the space L of piecewise-constant functions on $[0, 1]$ continuous from the left with the norm $\|f\| = \sup_{0 \le x \le 1} |f(x)|$. Describe the completion of L as a space of functions.

10. Let D be the space of continuously differentiable functions on $[0, 1]$ and $B \subset D$ be the set of continuously differentiable functions with absolute value of the derivative bounded by one with the uniform distance $d(f, g) = \max |f(x) - g(x)|$.

(1) Prove that D is dense in the space $C[0, 1]$ of continuous functions with the maximum norm.

(2) Prove that B is precompact.