MASS-11; ANALYSIS

FALL 2011

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HOMEWORK # 7

Due on Wednesday October 26

PROBLEMS 28 AND 30 HAVE BEEN CORRECTED

28. Let K be a non-empty closed convex subset of a Hilbert space H (real or complex). Prove that for any $x \in H$ there exist a unique point $y \in K$ closest to K.

29. Let L be a closed linear subspace of a real Hilbert space X. Prove that for any $x \in H$ the angle between x and $y \in L, \|y\| = 1$ reaches its minimum at a unique point y.

30. Let L_1 be a closed linear subspace and L_2 a finite-dimensional subspace of a real Hilbert space H. Prove that the angle between non-zero vectors $x \in L_1$ and $y \in L_2$ reaches its minimum and that minimum is positive if and only if $L_1 \cap L_2 = \{0\}$.

31. Find expansion coefficients with respect to the trigonometric basis $\exp 2\pi inx, n \in \mathbb{Z}$ in the space $L^2([0,1])$ for the following functions:

(1) $\exp ax, a \in \mathbb{R}$
(2) $\operatorname{sgn}(2x-1)$
(3) $x \exp x$.

32. Let m be a natural number and $\phi_m(x) = (-1)^{[2^m x]}$ $0 \le x \le 1$. Let $m_1 < m_2 < \cdots < m_n$. Prove that the the constant function equal to one and the functions

$$\phi_{m_1} \cdot \phi_{m_2} \cdot \ldots \cdot \phi_{m_n}$$

for different collections $m_1, \ldots, m_n$ form an orthonormal basis in $L^2([0,1])$ (real or complex). Here $[\cdot]$ denotes the integral part of the number.

33. Consider the following complex inner product in the space of polynomials of one complex variable $z = x + iy$:

$$(P,Q) = \int\int_{|z|\le 1} P(z)\overline{Q(z)}dxdy$$

Apply the orthogonalization process to the sequence of monomials $1, z, z^2, \ldots$ and find the corresponding orthonormal basis in that Hilbert space.