MASS-11; ANALYSIS

FALL 2011

A.Katok

HOMEWORK # 8

Due on Wednesday November 2

34. Consider the following scalar product in the space of trigonometric polynomials If $\chi_n = \exp 2\pi inx$ then

$$(\sum_{n=-N}^{N} a_n\chi_n, \sum_{n=-N}^{N} b_n\chi_n) = \sum_{n=-N}^{N} n^4 a_n \bar{b}_n.$$

(1) Prove that every Cauchy sequence in the norm generated by this inner product uniformly converges to a continuous function f.
(2) Express the functional $\ell(f) = f(0)$ in the completion of the space of trigonometric polynomials with respect to this norm as the inner product (f, f_0).

35. Give a description of the space dual to the space of continuous functions on the Cantor set, i.e. prove a counterpart of the Riesz Representation Theorem for that space.

Represent the Cantor set as the product of countable many copies of the two-point set and as the set of p-adic integers, and interpret your description in terms of these representations.

36. Prove that the weak topology in any infinite-dimensional normed space does not coincide with the strong (norm) topology.

37. Suppose that a sequence x_n of elements in a Hilbert space converges weakly to x. Prove that x_n converges in the strong topology is and only if $\lim_{n\to\infty} \|x_n\| = \|x\|$.

38. Prove that the space P of all polynomials cannot be made a Banach space, i.e. that there does not exist a norm in P such that P is complete with respect to this norm.

Hint: Use Baire Category Theorem.

EXTRA CREDIT PROBLEMS
(Due November 18; before the Thanksgiving break)

Solution of any two of the problems below may serve as the basis of the presentation of the research project in analysis.

6*. Replace n^4 in the definition of the inner product in Problem 34 with $|n|^a$ where a is a positive real number.

(1) For which values of a the assertion of the Problem 34 remains true?
(2) For which values of a one can assert that the limit is a differentiable function?

Try to come as close as possible to necessary and sufficient conditions.

7* Consider the projection of the infinite product of countably many copies of $\{0, 1\}$ to the unit interval given by the representation of real numbers in base 2.

Describe precisely the relation between the dual spaces to the continuous functions on the Cantor set (Problem 35) and on the interval.

8*. Prove that a Banach space is reflexive if and only if its unit ball is weakly compact.

9*. Find the distance for the point $x^n \in C([-1, 1])$ to the subspace of all polynomials of degree less than n.