FINAL EXAM TOPICS

SPACES: FROM ANALYSIS TO GEOMETRY AND BACK

The list below contains (i) concepts, notions, examples you must be familiar and comfortable with (including some basic properties of those), and (ii) results that will appear (possibly in a somewhat modified form) as "theoretical" items in the final examination questions . Items in the list are of unequal size and are in no direct connection with actual ticket questions. The order in this list does not coincide with the actual order of presentation in the lectures or in the notes

1. Topological spaces. Base of topology. Hausdorff property.

2. Metric spaces. Topology determined by metric.

3. Complete metric spaces. Existence and uniqueness of completion. Non-Archimedean (p-adic) completion of rationals. Completeness of the spaces of continuous functions on a compact space.

4. Baire Category theorem. Notions of a meager (first category) and residual set. Approximation of irrational numbers by rationals.

5. Normed spaces. Equivalence of norms in finite-dimensional spaces.

6. Euclidean spaces (finite-dimensional) over $\mathbb{R}$ and $\mathbb{C}$. Orthogonalization. Existence of orthonormal bases. Isometry of Euclidean spaces of a given dimension.

7. Convex bodies in a finite-dimensional linear space. Minkowski functional. Dual space for a finite-dimensional normed space. Extension theorem for linear functionals. Separation theorem for convex sets. Isometry of a space and its second dual.

8. Spaces ℓ^p in finite and infinite dimension. Hoelder and Minkowski inequalities. Completeness. Dual space to ℓ^p, $p > 1$.

9. Extreme points for compact convex sets in finite-dimensional spaces. Existence of extreme points.

10. Theorem: Every point in a compact convex set in $\mathbb{R}^n$ is a barycenter of a simplex with vertices in extreme points.

11. Convex polyhedra in $\mathbb{R}^n$. Vertices and faces of various dimensions. Theorem: A compact convex set is a finite intersection of half-spaces if and only if it has finitely many extreme points.

12. Banach spaces. Equivalence of boundedness and continuity for linear functions. Completeness of the dual space to a normed space. Polyhedral norms and duality.

13. Hahn-Banach theorem and its corollaries. Isometric embedding of a normed space into its second dual. Reflexivity.

14. Functions of bounded variation. Riemann-Stiltjes Integrals. Riesz Representation Theorem.

15. Stone-Weierstrass Theorem.

16. Hilbert spaces over $\mathbb{R}$ and $\mathbb{C}$. Orthogonality. Orthogonal projectios. Existence of orthonormal systems. Theorem: Every separable Hilbert space is isometric to ℓ^2.

17. L^2 as the completion of space of continuous functions. Characters form an orthonormal basis in $L^2(S^1)$. Fourier series.

18. Applications of the Baire category theorem to functional spaces. Nowhere differentiable functions.

19. Compactness criteria in the spaces of continuous functions.

20. Weak and weak –* topologies in Banach spaces. Theorem: every weakly bounded set is bounded.

21. Weak–* compactness of the closed balls in the dual to a separable normed space.

22. Extreme points for convex closed sets on Banach spaces. Existence of extreme points for a closed ball in the dual to a separable normed space.

23. Krein-Milman theorem.