Polynomial growth in nilpotent groups

THEOREM Any finitely generated nilpotent group has polynomial growth.

The proof goes by induction in nilpotent length.

The base of induction is nilpotent length one, i.e. abelian groups.

We start with the case of nilpotent length two to explain the ideas and consider a simple but important example. After that we present the general induction step.

Groups of nilpotent length two. Assume G is such a group with m generators, $g_1, \dots, g_m$. Then the commutator $[G,G]$ is abelian and belongs to the center of G. Take a product of n generators. Exchanging any two generators produces a commutator on the right. Since commutators lie in the center they can be moved to the right. In order to move generators to a canonical order one needs thus no more than n^2 interchanges. Thus we obtain a word of the form $g_1^{k_1} g_2^{k_2} \dots g_m^{k_m} C$, where C is the product of no more that n^2 commutators of the generators. Those commutators are words of bounded length with respect to any system of generators in the abelian group $[G,G]$ with polynomial growth of degree, say, k. Thus the growth of G is polynomial of degree at most $m+2k$.

EXAMPLE 1: Let Γ_3 be the group of upper-triangular unipotent 3×3 matrices with integer entries. Two generators,

$$e = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

one dimensional center generated by $[e,f] = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$; growth n^4. This is the slowest possible growth in a nilpotent group which does not have an abelian subgroup of finite index. It is a co-compact ltattice in the three-dimensional group of real unipotent upper-triangular matrices and the homogeneous space is thus a three-dimensional compact manifold (a *nil-manifold*) for which Γ is the fundamental group.

The general case. Assume G has nilpotent length s and as before has m generators $g_1, \dots, g_m$. Then $[G,G]$ has nilpotent length $\le s-1$ and hence by the inductive assumption has polynomial growth of degree, say k.

As before, consider a product w of n generators and try to bring it to a form $g_1^{k_1} g_2^{k_2} \dots g_m^{k_m} C$, where $C \in [G,G]$ and estimate the length of C. Exchanging a pair of generators produces a commutator on the right; as before there will be no more than n^2 such commutators in the process of rearranging the generators. But this time when we move generators to the left we need to exchange them with the commutators thus producing elements of the from $[g_{i_1}, [g_{i_2}, g_{i_3}]] \in [G,[G,G]]$, the total of no more of n^3, and so on. Since G has nilpotent length s this process of generating new terms will stop at s-th level, i.e. moving generators through commutators of i-th order with not produce any new terms. Thus the total length of C is estimated from above by $\mathit{const} \cdot n^s$ since there are at most $n^2 + \dots + n^s$ commutators of different orders and each of them is a word of bounded length. Thus the growth of G is polynomial of degree at most $m + sk$. □

The above prove give an above estimate of the degree of growth of G.

EXAMPLE 2: Consider the group Γ_4 of of upper-triangular unipotent 4×4 matrices with integer entries. It has three generators

$$e_{12} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; e_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } e_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

its nilpotent length is three and its commutant is isomorphic to Γ_3. This gives $3 + 3 \cdot 4 = 15$ as an above estimate on the degree of of growth for Γ_4. However, a more accurate count shows that the actual degree is 8.