GEOMETRIC STRUCTURES, SYMMETRY AND ELEMENTS OF LIE GROUPS

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1. Syllabus of the Course

− Groups, subgroups, normal subgroups, homomorphisms. Conjugacy of elements and subgroups.

− Group of transformations; permutation groups. Representation of finite groups as permutations.

− Group of isometries of the Euclidean plane. Classification of direct and opposite isometries. Classification of finite subgroups. Discrete infinite subgroups. Crystallographic restrictions

− Classification of similarities of the Euclidean plane.

− Group of affine transformations of the plane. Classification of affine maps with fixed points and connection with linear ODE. Preservation of ratio of areas. Pick's theorem.

− Group of isometries of Euclidean space. Classification of direct and opposite isometries.

− Spherical geometry and elliptic plane. Area formula. Platonic solids and classification of finite group of isometries of the sphere.

− Projective line and projective plane. Groups of projective transformations. Connections with affine and elliptic geometry.

− Hyperbolic plane. Models in the hyperboloid, the disc and half-plane. Riemannian metric. Classification of hyperbolic isometries. Intersecting, parallel and ultraparallel lines. Circles, horocycles and equidistants. Area formula.

− Three-dimensional hyperbolic space. Riemannian metric and isometry group in the upper half-space model. Conformal Möbius geometry on the sphere.

− Matrix exponential. Linear Lie groups. Lie algebras. Subalgebras, ideals, simple, nilpotent and solvable Lie algebras. Trace, determinant and exponential. Lie Algebras of groups connected to geometries studied in the course.

MASS lecture course, Fall 1999 .

2. Contents (Lecture by Lecture)

Lecture 1 (August 25)
Groups: definition, examples of noncommutativity (the symmetry group of the square D_4, $\mathrm{SL}(2, R)$), finite, countable (including finitely generated), continuous groups. Examples.

Lecture 2 (August 27)
Homomorphism, isomorphism. Subgroups. Generators. Normal subgroups, factors. Left and right actions.

Lecture 3 (August 30)
Conjugacy of elements and subgroups. Permutation groups. Representation of a finite group by permutations. Representation of any permutation as products of cycles and involutions.

Lectures 4-5 (September 3-5)
Groups of isometries of the Euclidean plane. Direct and opposite isometries. Theorem: any isometry is determined by images of three points not on a line. Theorem: a point and a ray determine exactly one direct and one opposite isometry. Groups of matrices. Representation of plane isometries by 3×3 matrices.

Lecture 6 (September 8)
Review: matrix description of isometries. Explicit description of SO(2) and O(2). Automorphisms and inner automorphisms in groups. Classes of isometries: translation, rotation (with matrix representation), reflection, glide reflection.

Lecture 7. (September 10)
Matrix representation of reflections and glide reflections. Conjugacy in the group of Euclidean isometries. Classification of isometries (two approaches: from the matrix description and as products of reflections).

Lecture 8 (September 13)
Semi-direct product structure and conjugacy classes in the group $\mathrm{Iso}(\mathbb{R}^2)$ of isometries of the Euclidean plane.

Lecture 9 (September 15)
Notion of discrete subgroup (in $\mathrm{Iso}(\mathbb{R}^2)$). Classification of finite subgroups (finish the "two center argument" next time). Crystallographic restrictions.

Lectures 10, 11 (September 17, 20)
End of proof: Any discrete subgroup of isometries is the semi-direct product of translations and a finite subgroup satisfying crystallographic restrictions. Different geometries.

Every discrete group has no more than two independent translations. Similarities, isometries of $\mathbb{R}^3$ and S^2. Euclidean geometry is about properties invariant under similarities. Geometries without nontrivial similarities.

Lecture 12 (September 22)
Description of similarities. Three approaches: synthetic (preservation of ratios of distances), linear algebraic, and complex (linear and anti-linear). Classification of

similarities: direct translations and spiral similarities; opposite glide reflections and dilative reflections. One-parameter groups of spiral similarities and focus for linear ODE.

Affine maps. Synthetic description (preservation of ratio of distances on a line) and linear algebraic. Proof of equivalence. Theorem: any bijection that takes lines into lines is affine.

Lecture 13 (September 24)
Invariance of the ratios of areas under affine maps (complete proof). Barycentric coordinates. Application to convex analysis.

Lecture 14 (September 27)
Lattices. Fundamental parallelograms. Area of fundamental parallelogram. Pick's Theorem. Beginning of discussion of the structure of affine maps on the plane. Fixed point, eigenvalues.

Lecture 15, (September 29)
Classification of affine maps with a fixed point. One parameter subgroups and ODE: node, saddle, linear shear, irregular node (focus missed; repeat next time). Inverted saddles and other opposite affine maps. Maps without fixed points.

Midterm exam: content and structure quickly discussed.

Lecture 16 (October 1)
One-parameter subgroups and matrix exponential. Preliminary discussion. Matrix norms and convergence. Isometries of $\mathbb{R}^3$. Four points define isometry (beginning of the proof).

Lecture 17 (October 4)
Isometries of $\mathbb{R}^3$ and S^2: first steps of their study.

Lecture 18 (October 6)
Isometries in $\mathbb{R}^3$. Classification.

Lecture 19 (October 13)
Spherical geometry. Comparison and contrast with Euclidean. Unique direct and opposite isometry taking a point and a ray into a point and a ray. The two first criteria of equality of triangles. Formula for the area (unfinished).

Lecture 20 (October 15)
End of the proof of the area formula. Equality of triangles with equal angles. Intersection of midpoint perpendiculars and medians. Finite groups of isometries.

Lecture 21 (October 18)
Structure of SO(3) as a three-dimensional projective space. Discussion of lower-dimensional situations, as well as of the one-dimensional complex projective space as the Riemann sphere.

Lecture 22 (October 20)
Finite group of isometries. Existence of a fixed point. Discussion of all examples except for icosahedron.

Additional assignment. In Coxeter's book [3] read about Platonic bodies and the classification of finite isometry groups.

Lecture 23 (October 25)
Preparation for the classification of finite isometry groups in three-dimensional space. Conjugate subgroups. Inner and outer automorphisms. Examples of outer automorphisms for $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{Z}^2$.

Lecture 24 (October 27)
Classification of regular polyhedra and regular tessellations on the plane. Proof based on the area formula (rather than the Euler Theorem). Classification of finite subgroups. Detailed discussion of the tetrahedron, sketch for the cube.

Lecture 25 (October 29)
(*Lecture by A. Windsor.*) First elements of hyperbolic geometry.

Lecture 26 (November 1)
Conclusion of the analysis of the uniqueness of the symmetry group of the cube. Hyperbolic geometry. Historical background.

Lecture 27 (November 3)
General concepts of geometry after Klein and Riemann. Examples of Riemannian manifolds (surfaces in $\mathbb{R}^3$, flat torus). Upper half plane with Poincaré metric as a Riemannian manifold. Isometry group.

Lecture 28 (November 5)
The group $\mathrm{SL}(2, \mathbb{C})$ and conformal geometry of the Riemann sphere. Line and circles mapped into lines and circles. Invariance of the cross-ratio. Existence and uniqueness of the Möbius transformation mapping three points into three points. Calculation of the distance in the hyperbolic plane via the cross-ratio.

Lecture 29 (November 8)
Structure of isometries on the hyperbolic plane: rotations (elliptic), parabolic, hyperbolic.

Lecture 30 (November 10)
Circles, horocycles, equidistants and more about one-parameter groups of isometries. Asymptotic triangles.

Lecture 31 (November 12)
(*Lecture by A. Windsor.*) The Gauss-Bonnet theorem for hyperbolic space of constant -1 curvature. Lemma: the group $\mathrm{PSL}(2, \mathbb{R})$ acts transitively on the unit tangent bundle and hence transitively on the space of geodesic arcs. This lemma is used to prove the fact that every isometry is either an element of $\mathrm{PSL}(2, \mathbb{R})$ or the composition of an element of $\mathrm{PSL}(2, \mathbb{R})$ with the standard reflection.

Lectures 32-33 (November 15,17)
Hyperbolic three-space in the upper-half space model. Metric and group (products of inversions) approach. General Riemannian metrics. Preservation of the hyperbolic metric by inversion. Horospheres and Euclidean geometry. Inversion is conformal.

Lecture 34 (November 19)
Inversion maps spheres into spheres. Spheres, horospheres and equidistant surfaces. Isometry is determined by four points.

Lecture 35 (November 22)
Matrix functions. Differentiation rules. Matrix algebras. Norms. Power series. Criterion of convergence. Matrix exponential. One-parameter subgroups and exponential. Differential equations. Lie algebras as tangent spaces.

*Lecture*s 36–39 (end of November)
Brackets (as degree of noncommutativity). Uniqueness of one-parameter subgroups. Trace, determinant, and exponential. Lie algebras of some classical groups. Campbell–Hausdorff formula (no proof). Adjoint representation. Ideals, simple, nilpotent and solvable Lie algebras. Main examples.

3. Homework Assignments

Problem Set # 1; August 27, 1999
(due on Wednesday September 1)

1. BML[1] p.126 N3.

2. BML p.131 N9.

3. BML p.136 N1.

4. BML p.136 N5.

5. BML p.137 N11.

6. BML p.140 N4.

7. BML p.153 N9.

8. BML p.157 N2.

9. Write a system of generators and relations for the group of symmetries of the set of integers on the real axis considered as a subset of the plane $\mathbb{R}^2$.

10. Write a system of generators and relations for the group of 2×2 matrices with entries in the field of two elements and with determinant one.

Additional (Extra Credit) Problems

A1. Find a free subgroup with two generators in the group $SL(2, R)$ of all 2×2 matrices with real entries and and with determinant one.

A2. The group of isometries of the Euclidean plane does not contain a free subgroup with two generators.

Problem Set # 2; September 2
(due on Thursday September 9)

11. Prove that the group of transformations of the extended real line $\mathbb{R} \cup \{\infty\}$ generated by the transformations $x \to 1/x$ and $x \to 1 - x$ is isomorphic to S_3.

12. BML p. 136 N2.

[1]BML stands for the Birkhoff MacLane textbook, see [1].

13. BML p. 139 N1.

14. BML p. 140 N2.

15. BML p. 140 N3.

16. BLM p. 146 N10.

Problem Set # 3; September 8
(due on Wednesday September 15)

17. Write the rotation by the angle $\pi/6$ around the point $(2, -5)$ in matrix form.

18. Write all glide reflections with the axis $x + 2y = 3$ in matrix form.

19. Coxeter, Section 3.4 (p.45)

20. Find all isometries (direct and opposite) which commute with a given translation.

21. Write all isometries commuting with the translation $T_{(2,-3)}$ in matrix form.

22. Describe a convex polygon whose full symmetry group is the cyclic group C_8. Make a careful drawing, preferably using ruler and compass.

23. Let $A = \begin{pmatrix} \cos 1 & \sin 1 \\ -\sin 1 & \cos 1 \end{pmatrix}$ and $v_0 = (3/8, -\sqrt{2}34)$. Let $Tv = Av + v_0$. Prove that T^{12345} is not a parallel translation.

24. Let A be an orthogonal 3×3 matrix, i.e., a matrix defining a Euclidean isometry in three–dimensional space. Assume that all eigenvalues of A are real numbers and let $\Lambda(A)$ be the set of eigenvalues of A (also called its *spectrum*). List all possible sets $\Lambda(A)$.

Additional (Extra Credit) Problems

A3. Given a glide reflection G, describe the collection of *all* ordered triples of lines such that the product of reflections in these lines equals G.

A4. Let T be a transformation of the plane which is *additive*, i.e., such that $T(v + w) = Tv + Tw$ for any vectors v, w. Is T necessarily linear?

Problem Set # 4, September 15
(due on Wednesday, September 22)

25-26. (Equivalent to two problems) Fill the "multiplication table" for the group $\mathrm{Iso}(\mathbb{R}^2)$ (details were given in class on September 13).

27. Describe conjugacy classes in the groups $\mathrm{Iso}(\mathbb{R}^2)$, and $\mathrm{Iso}_+(\mathbb{R}^2)$(details were given in class on September 13).

28. Consider the group $\mathrm{Aff}(\mathbb{R})$ of *affine* transformations of the real line, i.e., transformations of the form $x \to ax + b$, $a \neq 0$. Prove that translations form a normal subgroup in $\mathrm{Aff}(\mathbb{R})$ and that $\mathrm{Aff}(\mathbb{R})$ is the semi-direct product of this group and the group of homotheties $x \to ax$.

29. Prove that $\mathrm{Aff}(\mathbb{R})$ is exactly the group of all transformations of the real line preserving the *simple ratio* of three points

$$r(x_1, x_2, x_3) = \frac{x_1 - x_3}{x_2 - x_3}.$$

30. Prove that the group $\mathrm{Sim}(\mathbb{R}^2)$ of *similarities* of the plane defined as the group of all transformations which preserve the ratio of lengths of any two intervals can be equivalently defined as the group of all affine transformations which preserve angles between any pair of lines.

31. Prove that any similarity transformation which does not have a fixed point is an isometry.

32. Prove that any similarity transformation with a fixed point is the product of a homothety H and a rotation or reflection which commutes with H.

Additional (Extra Credit) Problem

A4. The group $GL(2,\mathbb{R})$ of 2×2 matrices with determinant different from zero acts on the unit circle S^1 as follows:

for $(x,y),\ x^2+y^2=1$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R})$, let

$$f_A(x,y) = \left(\frac{ax+by}{((ax+by)^2+(cx+dy)^2)^{1/2}}, \frac{cx+dy}{((ax+by)^2+(cx+dy)^2)^{1/2}} \right)$$

Find a description of the corresponding geometry on the circle. More specifically, find a quantity depending on *four* points whose preservation characterizes the transformations $f_A, A \in GL(2,\mathbb{R})$ among all bijections of the circle.

Problem Set # 5 September 22
(due on Wednesday September 29)

33. Describe the group of isometries of the plane with the metric

$$d_{\max}((x_1,y_1),(x_2,y_2)) := \max(|x_2-x_1|, |y_2-y_1|).$$

Give complete proofs.

34. Identify the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by putting $x+iy=z$. Prove that direct (orientation-preserving) similarities of $\mathbb{R}^2$ are represented as linear transformations $z \to az+b$, where $a,b\in\mathbb{C}$ and $a\neq 0$ and opposite (orientation–reversing) similarities become anti–linear transformations $z\to a\bar{z}+b$.

35. Describe conjugacy classes in the group $\mathrm{Sim}(\mathbb{R}^2)$ in two ways: (i) using the representation from the previous problem and, (ii) in geometric terms (direct: translations, rotations, spiral similarities; opposite: reflections, glide reflections, dilative reflections). Specific questions: Is it true that the conjugacy class of a rotation in $\mathrm{Sim}(\mathbb{R}^2)$ coincides with its conjugacy class in $\mathrm{Iso}(\mathbb{R}^2)$? The same question for translations.

36. The factor–group $\mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{Z}^2$ is the lattice of all vectors with integer coordinates, is called the *torus* and is denoted by T^2. The (Euclidean) distance on the torus is defined as the minimum of Euclidean distance between the elements of corresponding cosets. Describe the group of isometries of the torus with Euclidean distance.

37. Prove that any direct isometry in the three-dimensional Euclidean space with a fixed point is a rotation around an axis passing through this point.

38. Prove that any normal subgroup of $\mathrm{Sim}(\mathbb{R}^2)$ contains all translations.

39. How many affine transformations map one given triangle (without marked vertices) into another?

40. Given two quadrangles $ABCD$ and $A'B'C'D'$, describe a procedure allowing to determine whether there exists an affine transformation which maps one into the other.

Additional (Extra Credit) Problems

A5. Prove that any normal subgroup H of $\mathrm{Iso}(\mathbb{R}^2)$ contains all translations. *Note:* The subgroup H is not assumed to be closed.

A6. Prove that the sum of angles of a triangle on the sphere formed by arcs of great circles is always greater than π.

Problem Set # 6, September 29
(due on Wednesday October 13)

41. Prove that any isometry of the sphere with a fixed point is a product of at most two reflections.

Give an example of an isometry of the sphere which cannot be represented as a product of one or two reflections.

42. Prove that any rotation of the sphere is the product of two half-turns (rotations by π).

43. The *elliptic plane* $\mathbb{E}^2$ is obtained from the sphere S^2 by identifying pairs of opposite points. The distance between the pairs $(x, -x)$ and $(y, -y)$ is defined as $\min(d(x, y), d(x, -y))$, where d is the distance on the sphere (the angular distance). Lines on the elliptic plane are defined as images of great circles on the sphere under identifications. Thus any two lines on the elliptic plane intersect at exactly one point.

Prove that the group of isometries of elliptic plane, $\mathrm{Iso}(\mathbb{E}^2)$, is isomorphic to the group $\mathrm{SO}(3, \mathbb{R})$ of direct isometries of the sphere and hence is connected.

44. Prove that any isometry of $\mathbb{E}^2$ has a fixed point.

45. Show that on the elliptic plane one can define rotations around a point and reflections in a line. Prove that any reflection in a line coincides with the half–turn around a certain point.

46. There are two natural ways to define the distance between points in spherical geometry: as Euclidean distance inherited from three-dimensional space and the angular distance.

Find reasons why the second definition is preferable for the purposes of internal geometry.

47. Prove that any opposite (orientation- reversing) affine map of the plane without fixed points is conjugate to a map $L_\lambda : L_\lambda(x,y) = (x+1, -\lambda y)$ for some $\lambda > 0$.

48. Read about Farey series in Coxeter, Section 13.5. Solve Problem 2 in that section: If $y_0/x_0, y/x, y_1/x_1$ are three consecutive terms in the Farey series, then

$$\frac{y_0 + y_1}{x_0 + x_1} = \frac{y}{x}.$$

Additional (Extra Credit) Problems

A7. Consider the group of all bijections of the elliptic plane (not necessarily isometries) which map lines into lines and preserve a particular line l. Prove that this group is isomorphic to $\mathrm{Aff}(\mathbb{R}^2)$.

A8. Coxeter Section 13.5, Problem N 7.

Problem Set # 7, October 13
(due on Wednesday October 20)

49. Given positive numbers a, b, and c, find necessary and sufficient conditions for the existence of a triangle on the sphere of radius 1 (or the corresponding elliptic plane) with sides of length a, b and c.

50. Given numbers α, β, and γ between 0 and π, find necessary and sufficient conditions for the existence of a triangle on the elliptic plane with interior angles α, β, and γ.

51. Calculate the length of the sides of an equilateral triangle with summit angle α on the sphere of radius 1.

52. Prove that bisectors of the angles in a spherical triangle intersect in a point inside the triangle.

53. Prove the equality of triangles on the sphere with equal pairs of sides.

54. Prove that any finite group of affine transformations in $\mathbb{R}^n$ has a fixed point.
Hint: Use the notion of center of gravity.

55. Let P be convex polyhedron with 14 vertices

$$(\pm 1, \pm 1, \pm 1),\ (\pm 2, 0, 0)\ (0, \pm 2, 0),\ (0, 0, \pm 2).$$

Find the symmetry group of P.

Additional (Extra Credit) Problem

A9. Prove that the altitudes in the acute triangle on the sphere intersect at a point inside the triangle. Find a similar statement for other triangles.

Problem Set # 8, October 27
(due on Wednesday November 3)

56. Describe a *fundamental domain* for the full groups of isometries for
(a) the regular tetrahedron; (b) the cube .

57. Describe a convex fundamental domain for the group of rotations of the cube.

58. Describe a tiling of the three-dimensional space by isometric copies of

(a) a tetrahedron; (b) an octahedron.

59. Prove that rotations of the regular tetrahedron of order two (including the identity) form a normal subgroup in the group of all rotations of the regular tetrahedron.

60. Find a normal subgroup of order four in the group of isometries of the cube.

61. Coxeter, Section 15.4, N1

62. Describe a subgroup of rotations of the regular icosahedron isomorphic to the group A_4 of even permutations of four symbols.

63. Describe a polyhedron for which the pair (full isometry group, rotation group) is (C_{2n}, C_n).

Additional (Extra Credit) Problem

A10. Prove that any closed infinite subgroup of SO(3) contains a all rotations around a certain axis.

A11. Prove that the three dimensional space cannot be tiled by isometric copies of

(a) a regular tetrahedron; (b) a regular octahedron.

Hint: Use the results of Section 10.4 in Coxeter. You may also use calculator to check certain inequalities involving angles. However, you should justify your result.

A12. Prove that the group of rotations of the regular icosahedron is *simple*, i.e., does not have proper normal subgroups other than the identity.

Problem Set # 9, November 3
(due on Wednesday November 10)

64. Calculate the angle of parallelism in the Poincaré disc model of the hyperbolic plane $\mathbb{H}^2$ at the point $i/2$ with respect to the line represented by the real diameter: $\{-1 < t < 1\}$.

65. What statement in the hyperbolic geometry corresponds to the Euclidean statement : "product of two reflections in parallel axis is the parallel translation by a vector perpendicular to the axis."

66. Find the product of two half-turns (rotations by π) in $\mathbb{H}^2$.

67. (i) Prove that in the Poincaré upper half-plane model the center of any circle in hyperbolic geometry is different from its Euclidean center.

(ii) Find all circles in the Poincaré disc model for which the hyperbolic center coincides with the Euclidean center.

68. Prove that bisectors of three angles in a hyperbolic triangle intersect in a single point.

69. What statement in hyperbolic geometry corresponds to the Euclidean statement: "if three points do not lie on a line, then there exists a unique point equidistant from all three points which is the center of the circle passing through these points."

Additional (Extra Credit) Problem

A13. Prove that any one-to-one continuous transformation of $\mathbb{H}^2$ which maps hyperbolic lines into hyperbolic lines is an isometry. Thus there is no separate affine hyperbolic geometry!

Problem Set # 10, November 10
(due on Wednesday, November 17)

70. Prove that any parallel translation (hyperbolic isometry) on the hyperbolic plane belongs to two different two-dimensional subgroups of the group of direct isometries $\mathrm{Iso}(\mathbb{H}^2)$ (which is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$).

71. Find the side of the quadrilateral (4-gon) with four angles equal $\pi/4$ each.

72. Prove that for any two lines l and l' in the hyperbolic plane the locus of points equidistant from l and l' is a line .

73. Suppose lines l and l' are ultraparallel. Then the equidistant line from the previous problem can be constructed as follows: there is a unique pair of points $p \in l,\ p' \in l'$ such that $\mathrm{dist}(p,p') \le \mathrm{dist}(q,q')$ for all $q \in l,\ q' \in l'$. Let s be the midpoint of the interval (p,p'). The equidistant line is the perpendicular to the line pp' at s.

74. For any two pairs of parallel lines $l,\ l'$ and $l_1,\ l_1'$ there exists an isometry T such that $T(l) = l_1$ and $T(l') = l_1'$.

75. Prove that there are exactly two horocycles passing through two given points.

76. Calculate the length of a horocycle segment between two points at the distance d (in particular, show that two such segments have equal length).

77. Calculate the length of a circle of radius r and the area of a disc of radius r in the hyperbolic plane.
Hint: Use the Poincaré unit disc model, and place the center at the origin.

Problem Set # 11, November 17
(due on Wednesday November 24 (before Thanksgiving))

78. Given a point $p \in \mathbb{H}^3$ (the three-dimensional hyperbolic space) and an ordered pair of perpendicular rays (half-lines) r_1, r_1' at this point, there exists exactly one direct and one opposite isometry which takes the configuration p, r_1, r_1' into another given configuration of the same kind.

79. If an isometry of $\mathbb{H}^3$ has two fixed points, then it has infinitely many.

80. Prove that any direct isometry in $\mathbb{H}^3$ with a fixed point is a rotation around a certain axis by a certain angle.

81. A direct isometry of $\mathbb{H}^3$ is called *parabolic* if it has a unique fixed point at the sphere at infinity. Classify parabolic isometries up to a conjugacy in the group $\mathrm{SL}(2,\mathbb{C})$.

82. Prove that the product of reflections in two ultraparallel planes in $\mathbb{H}^3$ has a unique invariant line. The product is called a *translation* and the invariant line is called its *axis*.

83. A direct isometry of $\mathbb{H}^3$ is called *loxodromic* if it is not a rotation or a translation. Prove that any loxodromic isometry has a unique invariant line (called *the axis*) and is the composition of a translation along the axis and a rotation around it. Show that the translation along a line and the rotation around the same line commute.

84. Consider the natural embedding of $\mathrm{SL}(2,\mathbb{C})$ into $\mathrm{GL}(4,\mathbb{R})$ obtained by replacing the complex number $x+iy$ by the 2×2 matrix $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$. Find the Lie algebra of this group.

85. Consider the natural embedding of the group $\mathrm{Aff}\,(\mathbb{R}^2)$ into $\mathrm{GL}(3,\mathbb{R})$ obtained by identifying the affine map $x \to Ax+b$ with the matrix $\begin{pmatrix} A & b \\ \mathbf{0} & 1 \end{pmatrix}$. Find the Lie algebra of this group.

86. A subalgebra $\mathfrak{a}$ of a Lie algebra $\mathfrak{l}$ is called an ideal if $[\mathfrak{l},\mathfrak{a}] \subset \mathfrak{a}$. Prove that the Lie algebra $\mathfrak{so}(3,\mathbb{R})$ of skew–symmetric 3×3 matrices is simple, i.e., has no ideals other than $\{0\}$ and itself.

Additional (Extra Credit) Problem

A14. Define a regular tessellation of $\mathbb{H}^3$ and construct an example.

Problem Set # 12, November 29
(due on Friday, December 3)

Note: Problems 84–86 from problem set # 11 are due on Monday November 29.

87. The group of upper-triangular 3×3 matrices with ones on the diagonal is called *the Heisenberg group*. Find all one–parameter subgroups in the Heisenberg group, its Lie algebra and write out an explicit formula for the exponential of an element of the Lie algebra.

88. Is the group $\mathrm{Aff}(\mathbb{R})$ of affine transformations of the real line (a) solvable; (b) nilpotent?

89. Consider the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of traceless 2×2 matrices. Characterize the exponential of this Lie algebra as a subset of the group $SL(2,\mathbb{R})$.
Hint: Consider eigenvalues.

90. Consider the special unitary group $\mathrm{SU}(2)$ of 2×2 complex matrices of the form $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, $a,b\in\mathbb{C}$, $|a|^2+|b|^2=1$ as a Lie group (this group is isomorphic to a subgroup of $\mathrm{SL}(4,\mathbb{R})$). Prove that its Lie algebra $\mathfrak{su}(2)$ consists of matrices $\begin{pmatrix} \alpha i & c \\ -\bar{c} & -\alpha i \end{pmatrix}$, $\alpha\in\mathbb{R}$, $c\in\mathbb{C}$.

Additional (Extra Credit) Problems

A15. Find a free subgroup with two generators in the group $\mathrm{SL}(2, R)$ of all 2×2 matrices with real entries and and with determinant one.

A16. The group of isometries of the Euclidean plane does not contain a free subgroup with two generators.

4. Topics for the Midterm Exam

This section consists of a list of the main theoretical topics of the first half of the course; this list was given to the students to help them prepare for the midterm examination.

Groups – general definitions. Transformation groups and permutations. Isometries of the Euclidean plane.

Finite and discrete subgroups of the isometry groups with no more than two independent translations and crystallographic restrictions.

Similarities of the Euclidean plane.

Affine transformations in the plane, including one-parameter subgroups.

Limited amount of isometries of the three-space and sphere (based on the material from HW up to 5).

5. Midterm Exam

The students were informed in advance about the format of the midterm, i.e., they were told that the exam would consist of:

1. *Two or maybe three problems.*
2. *One or two theoretical questions* (*theorems, definitions, examples*).
3. *Several questions requiring answers, but not detailed solutions.*

The rough distribution of weights among sections was: $5:2:3$ *or* $6:2:2$.

Part 1. Problems. *Detailed solutions required.*

1.1. May a discrete subgroup of $\mathrm{Iso}(\mathbb{R}^2)$ contain the rotation by $2\pi/3$ around the origin and rotation by $\pi/2$ about the point $(0, 1)$?

1.2. An affine transformation which is not an identity $L \in \mathrm{Aff}(R^2)$ satisfies the condition $L^7 = \mathrm{Id}$. Prove that L is conjugate in $\mathrm{Aff}(R^2)$ to one of three different non–conjugate rotations.

Part 2. Theoretical questions.

Detailed descriptions required; complete proofs are optional.

2.1. Describe how products of one, two, or three reflections represent different kinds of isometries in the plane.

2.2. Define a normal subgroup in a group. Give examples of nonnormal subgroups in (i) the group D_3 of symmetries of the equilateral triangle. (ii) the group $\mathrm{Aff}(\mathbb{R}^2)$ of affine transformations of the plane.

Part 3. Questions. *Give answers. Explanations are optional.*

3.1. Is it true that any two translations conjugate in the group $\mathrm{Aff}(\mathbb{R}^3)$ are conjugate in the group of similarities of $\mathrm{Sim}(\mathbb{R}^3)$?

3.2. Describe a nontrivial normal subgroup in $\mathrm{Aff}(\mathbb{R}^2)$ which contains all isometries.

3.3. A one-parameter group of affine transformations commutes with the one parameter group H_t of "hyperbolic rotations": $H_t(x,y) = (2^t x, 2^{-t} y)$. Is it possible that this subgroup represents: (i) a node; (ii) a focus?

6. Topics for the Final Examination

This section contains a list of the main topics covered in the course; this is the list given to the students to help them prepare for the final.

1. Definition of group, subgroup, homomorphism, isomorphism, normal subgroup, factor group; conjugacy of elements and subgroups.

Theorem: the kernel of a homomorphism is a normal subgroup.

Permutations.

Theorem: any finite group is isomorphic to a subgroup of a permutation group.

Representation of any permutation as the product of cycles and involutions.

2. Groups of isometries of the Euclidean plane. Direct and opposite isometries.

Theorem: any isometry is determined by images of three points not on a line.

Theorem: a point and a ray determine exactly one direct and one opposite isometry.

Representation of plane isometries by 3 by 3 matrices. Classification of plane isometries (four principal types and up to a conjugacy).

Classification of finite groups of isometries of the plane.

Discrete subgroups of the group of isometries of the plane.

Theorem: any discrete group contains no more than two translations independent over the rational numbers.

Crystallographic restrictions theorem.

3. Similarities of the line and the plane.

Three ways of characterizing similarities: synthetic (preservation of ratios of distances), linear algebraic and via complex numbers (linear and anti-linear).

Classification of similarities. Four principal types: direct–translations and spiral similarities; opposite glide reflections and dilative reflections. Classification up to conjugacy.

4. Group of affine transformations of the plane. Synthetic description (preservation of ratio of distances on a line) and linear algebraic description. Proof of equivalence.

Invariance of the ratios of areas under affine maps.

Lattices. Fundamental parallelograms. Area of fundamental parallelogram. Pick's Theorem.

One parameter subgroups of linear maps and ODE: node, saddle, linear shear, irregular node, focus.

5. Isometries in the three-dimensional Euclidean space.

Theorem: images of four points not in a plane determine an isometry.

Theorem: any isometry is a product of no more than four reflections.

Classification of isometries in three-dimensional Euclidean space. Main types and conjugacy invariants.

Isometries of the sphere.

Theorem: a point and a ray determine exactly one direct and one opposite isometry.

Three criteria of triangle equality in spherical geometry.

Area formula for spherical triangles.

Theorem: any finite group of isometries in the Euclidean space has a fixed point.

Rotations group of Platonic solids. Representation of the tetrahedron and the cube group as A_4 and S_4 permutation groups.

6. Projective line. Action of $\mathrm{SL}(2, \mathbb{R})$. Preservation of the cross-ratio. Projective and elliptic plane. Nonorientability. Action of $\mathrm{SL}(3, \mathbb{R})$ at the projective plane.

Affine plane as projective plane without line at infinity.

7. Hyperbolic plane. Models in the half-plane and disc. Riemannian metric.

Isometries of the hyperbolic plane. Preservation of Riemannian metric by fractional linear and anti-fractional linear transformations.

Preservation of the cross-ratio by fractional linear transformations.

Theorem: any transformation preserving cross-ratio is Möbius.

Theorem: every isometry of hyperbolic plane is determined by three points. Expression of distance through cross-ratio.

Classification of direct isometries of the hyperbolic plane. Elliptic, parabolic, and hyperbolic isometries.

Theorem: images of a point and a ray determine one direct and one opposite isometries.

Circles, horocycles and equidistants in the hyperbolic plane. Their representation in the half-plane and disc models.

Area formula for a triangle in the hyperbolic plane. Proper and asymptotic triangles.

8. Three-dimensional hyperbolic space. Metric and isometry group in the upper half-space model.

Preservation of hyperbolic metric by inversions.

Theorem: inversion maps planes and spheres into planes and spheres.

Theorem: every isometry of the hyperbolic space is a product of no more than four inversions.

Spheres, horospheres and equidistant surfaces in hyperbolic space.

9. Matrix exponential. One-parameters subgroups of $\mathrm{GL}(n, \mathbb{R})$ and exponentials.

Definition of linear Lie group. Lie algebra as tangent space at the identity. Invariance under taking brackets.

Definition of an abstract Lie algebra. Subalgebras, ideals, simple, nilpotent and solvable Lie algebras.

Trace, determinant and exponential. Lie algebras of $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SO}(n,\mathbb{R})$.

7. Final Examination Tickets

Examination ticket # 1.

1. *Give a complete proof of the following statement:* Any finite group is isomorphic to a subgroup of the permutation group.

2. *Problem:* Given a horocycle h in the hyperbolic plane and a point $p \in h$, how many other horocycles are there whose intersection with h consists of the single point p?

3. *Answer the following questions (proofs are optional):*

Any matrix $A \in \mathrm{SL}(2,\mathbb{R})$ belongs to a one-parameter subgroup. *Yes/No?*

Any direct isometry of the hyperbolic plane $\mathbb{H}^2$ belongs to a one-parameter subgroup of $\mathrm{Iso}\,(\mathbb{H}^2)$. *Yes/No?*

Examination ticket # 2.

1. *Give a complete proof of the following statement:* Any isometry of the Euclidean, elliptic or hyperbolic plane is determined by the images of three points not on a line.

2. *Problem:* A pentagon P in the hyperbolic plane lies inside a triangle and has all five angles equal to α. Prove that $\alpha > 2\pi/5$.

3. *Answer the following question (proofs are optional):* Does every isometry of the elliptic plane have a fixed point? *Yes/No?*

Examination ticket # 3

1. *Give a complete proof of the following:* The classification of finite groups of isometries of the plane.

2. *Problem:* An isometry T of the hyperbolic plane in the upper half–plane model maps the point i into $2i$ and leaves the point $1+i$ on the horocycle $\Im z = 1$. Find all possible values of $T(1+i)$.

3. *Answer the following question (proofs are optional):* What is the minimal dimension of a nonabelian nilpotent Lie algebra?

Examination ticket # 4.

1. *Give a complete proof of:* The crystallographic restrictions theorem.

2. *Problem:* Prove that all isometries of the hyperbolic plane which commute with a parabolic isometry are direct.

3. *Answer the following questions (proofs are optional):*

Is any projective transformation of the projective line with three fixed points the identity? *Yes/No?*

Is any affine transformation of the three–dimensional space with three fixed points the identity? *Yes/No?*

Examination ticket # 5.

1. *Give a complete proof of:* The classification of plane isometries (four principal types and classification up to conjugacy within the isometry group).

2. *Problem:* Find all parabolic isometries of the hyperbolic plane in the upper half–plane model which map the point $i+2$ into $i-1$.

3. *Answer the following question* (*proofs are optional*)*:* Does every isometry of elliptic plane belongs to a one-parameter group of isometries? *Yes/No?*

Examination ticket # 6.

1. *Give a complete proof of the following statement:* Any direct similarity of the plane is either a translation or a spiral similarity.

2. *Problem:* Prove that any direct isometry of the hyperbolic space $\mathbb{H}^3$ with a fixed point is a product of two reflections or inversions in the upper half-space model.

3. *Answer the following question* (*proofs are optional*)*:* Is any finite group of rotations in $\mathbb{R}^3$ with an odd number of elements cyclic? *Yes/No?*

Examination ticket # 7.

1. *Give a complete proof of the following statement:* Any opposite similarity of the plane is either a glide reflection or a dilative reflection.

2. *Problem:* Find all isometries of the hyperbolic plane in the upper half-plane model which map the line $\Im z = 1$ into the circle $|z-2| = 3$.

3. *Answer the following question* (*proofs are optional*)*:* Is the exponential of any matrix Lie algebra a Lie group? *Yes/No?*

Examination ticket # 8.

1. *Give a complete proof of the following statement:* Any discrete group of isometries of the plane contains no more than two translations independent over the rational numbers.

2. *Problem:* Consider two hyperbolic isometries of the hyperbolic plane whose axes are parallel. May their product be a rotation?

3. *Answer the following questions* (*proofs are optional*):

Do symmetric 3×3 matrices form a Lie algebra? *Yes/No?*

Do skew-symmetric 3×3 matrices form a Lie algebra? *Yes/No?*

Examination ticket # 9.

1. *Give a complete proof of* Pick's Theorem about the area of a polygon with vertices in a lattice.

2. *Problem:* Find all isometries of the hyperbolic plane which fix the horocycle $|z-i| = 1$.

3. *Answer the following question* (*proofs are optional*)*:* What is the minimal positive angle of rotation which belongs to a discrete group of isometries in the Euclidean plane?

Examination ticket # 10.

1. *Give a complete proof of the following statements:*

Images of four points not in a plane determine an isometry in Euclidean space.

Any isometry of Euclidean space is a product of no more than four reflections.

2. *Problem:* Let H be a half–turn in the plane. Find the centralizer $Z(H)$ in $\mathrm{Aff}(\mathbb{R}^2)$, i.e., the group of affine transformations commuting with H.

3. *Answer the following question* (*proofs are optional*)*:* Is any horocycle segment in the hyperbolic plane at most three times longer than the distance between its ends? *Yes/No?*

Examination ticket # 11.

1. *Give a complete proof of the* Area formula for spherical triangles.

2. *Problem:* Prove that any parabolic isometry of the hyperbolic plane belongs to a two-dimensional subgroup of $\mathrm{Iso}(\mathbb{H}^2)$.

3. *Answer the following question* (*proofs are optional*)*:* Do any four points in the hyperbolic space which do not lie on a line belong to a certain sphere? *Yes/No?*

Examination ticket # 12.

1. *Give a complete proof of the following statement:* Any finite group of isometries in the Euclidean space has a fixed point.

2. *Problem:* A hyperbolic isometry in the upper half-plane model maps the point $i-1$ into $i+1$. Describe all possible images of the point i.

3. *Answer the following questions* (*proofs are optional*)*:*

Is any affine transformation with a fixed point a similarity? *Yes/No?*

Is any similarity without a fixed point an isometry? *Yes/No?*

Examination ticket # 13.

1. *Give a complete proof of* The classification of direct isometries of the hyperbolic plane into elliptic, parabolic and hyperbolic isometries.

2. *Problem:* Find all normal subgroups of the group D_4 of isometries of the square.

3. *Answer the following questions* (*proofs are optional*)*:*

For a (nondegenerate) triangle in the hyperbolic plane, do the perpendicular bisectors of its sides intersect in a single point? *Yes/No?*

For a triangle in the elliptic plane, do the perpendicular bisectors of its sides intersect in a single point? *Yes/No?*

Examination ticket # 14.

1. *Give a complete and rigorous account of the following:* Define circles, horocycles and equidistants in the hyperbolic plane and describe how they are represented in the half-plane and disc models.

2. *Problem:* Let $ABCD$ be a quadrilateral in the plane and assume that no triple of its vertices lie on a line. Consider the group of affine transformations which map the quadrilateral into itself. What are possible numbers of elements in that group?

3. *Answer the following questions* (*proofs are optional*)*:*

Is any opposite isometry of the Euclidean plane without fixed points a glide reflection? *Yes/No?*

Is any opposite isometry of the hyperbolic plane with a fixed point a reflection? *Yes/No?*

Examination ticket # 15.

1. *Give a complete proof of the* Area formula for a triangle in the hyperbolic plane.

2. *Problem:* Prove that the product of reflections in the sides of a triangle in Euclidean plane is never a reflection.

3. *Answer the following question* (*proofs are optional*)*:*

Any three-dimensional Lie algebra is either abelian, or isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, or isomorphic to $\mathfrak{so}(3,\mathbb{R})$. *Yes/No?*

Examination ticket # 16.

1. *Give a complete proof of the following statements:*

Any transformation of the hyperbolic plane preserving cross-ratio is Möbius.

Every isometry of the hyperbolic plane is determined by three points. Express the distance via the cross-ratio.

2. *Problem:* Find a necessary and sufficient condition for two glide reflections in Euclidean or hyperbolic plane to commute.

3. *Answer the following question* (*proofs are optional*)*:* Any finite group of affine transformations in Euclidean space is conjugate to a to a subgroup of the isometry group. *Yes/No?*

Examination ticket # 17.

1. *Give complete proofs of the following statement:* Define the Riemannian metric in the upper half-plane model of the hyperbolic plane and show that any Möbius transformation preserves this metric.

2. *Problem:* Find a necessary and sufficient condition for a rotation and a spiral isometry in Euclidean space to commute.

3. *Answer the following questions* (*proofs are optional*)*:*

Is the group of affine transformations of the plane isomorphic to the group of projective transformations of the projective plane preserving a certain line? *Yes/No?*

Is the group of affine transformations of the line isomorphic to the group of isometries of the hyperbolic plane preserving a certain point at infinity? *Yes/No?*

Examination ticket # 18.

1. *Give a complete proof of the following statements:*

A point and a ray determine exactly one direct and one opposite isometry in the Euclidean plane or the hyperbolic one.

A point and a ray determine exactly one isometry in the elliptic plane.

2. *Problem:* Suppose T is a direct isometry of the hyperbolic plane, l and l' are non–parallel lines, Tl is parallel to l and Tl' is parallel to l'. Prove that T is a hyperbolic isometry.

3. *Answer the following questions* (*proofs are optional*)*:*

Does there exist a discrete infinite subgroup of the group of isometries of the Euclidean plane which contains no parallel translations? *Yes/No?*

Does there exist a discrete infinite subgroup of the group of isometries of the Euclidean space which contains no parallel translations? *Yes/No?*

Examination ticket # 19

1. *Give a complete proof of the following statement:* Every isometry of the hyperbolic space in the upper half–space model is a product of no more than four inversions.

2. *Problem:* Find the minimal number of generators in the group of rotations of the tetrahedron.

3. *Answer the following questions* (*proofs are optional*)*:*

Does any direct similarity of the Euclidean plane belong to a one-parameter subgroup? *Yes/No?*

Does any direct affine transformation of the plane belong to a one-parameter subgroup? *Yes/No?*

Examination ticket # 20.

1. *Give a complete proof of the following statement:* The Riemannian metric in the upper half-space model of the hyperbolic space is preserved by inversions.

2. *Problem:* A direct similarity S of the plane which is not an isometry and a parallel translation T, never commute.

3. *Answer the following question* (*proofs are optional*)*:* List all finite groups of rotations in three–dimensional space.

Examination ticket # 21.

1. *Give a complete proof of the following statement:* The tangent space to a linear Lie group is a Lie algebra, i.e., is invariant with respect to the bracket operation.

2. *Problem:* Find all isometries of the hyperbolic plane, direct and opposite, (in the upper half-plane model) that preserve the equidistant curve $\Im z = 2\Re z$.

3. *Answer the following question* (*proofs are optional*)*:* What is the minimal positive angle of rotation in a finite group G of rotations in $\mathbb{R}^3$ if it is known that G is neither cyclic, nor dihedral?

Examination ticket # 22.

1. *Give a rigorous account with complete proofs:* Describe Lie algebras of $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SO}(n,\mathbb{R})$.

2. *Problem:* A direct isometry T of the hyperbolic plane preserves two different equidistant curves e_1 and e_2. Prove that either T^2 is the identity or e_1 and e_2 are equidistant from the same line.

3. *Answer the following question* (*proofs are optional*)*:* Does every similarity in $\mathbb{R}^3$ which is not an isometry have a fixed point? *Yes/No?*

Examination ticket # 23.

1. *Give a complete proof of the following statements:*

Any transformation of the hyperbolic plane preserving cross-ratio is Möbius.

Every isometry of hyperbolic plane is determined by three points. Express the distance via the cross-ratio.

2. *Problem:* Find a necessary and sufficient condition for two glide reflections in Euclidean or hyperbolic plane to commute.

3. *Answer the following question* (*proofs are optional*)*:* Any finite group of affine transformations in the Euclidean space is conjugate to a group of isometries. *Yes/No?*

Examination ticket # 24.

1. *Give a rigorous account and complete proofs:* Define the Riemannian metric in the upper half-plane model of the hyperbolic plane and prove that any Möbius transformation preserves this metric.

2. *Problem:* Find a necessary and sufficient condition for a rotation and a spiral isometry in Euclidean space to commute.

3. *Answer the following questions* (*proofs are optional*)*:*

Is the group of affine transformation of the plane isomorphic to the group of the projective transformations of the projective plane that preserve a certain line? *Yes/No?*

The group of affine transformation of the line is isomorphic to the group of isometries of the hyperbolic plane preserving a certain point at infinity. *Yes/No?*

Examination ticket # 25.

1. *Give a complete proof of the following statements:*

A point and a ray determine exactly one direct and one opposite isometry in the Euclidean plane or in the hyperbolic plane.

A point and a ray determine exactly one isometry in the elliptic plane.

2. *Problem:* Suppose T is a direct isometry of the hyperbolic plane, l and l' are non-parallel lines, Tl is parallel to l and Tl' is parallel to l'. Prove that T is a hyperbolic isometry.

3. *Answer the following questions* (*proofs are optional*)*:*

Does there exist a discrete infinite subgroup of the group of isometries of the Euclidean plane which contains no parallel translations? *Yes/No?*

Does there exist a discrete infinite subgroup of the group of isometries of the Euclidean space which contains no parallel translations? *Yes/No?*

Examination ticket # 26.

1. *Give a complete proof of the following statement:* Every isometry of the hyperbolic space is the product of no more than four inversions.

2. *Problem:* Find the minimal number of generators in the group of rotations of the tetrahedron.

3. *Answer the following questions* (*proofs are optional*)*:*

Does any direct similarity of the Euclidean plane belong to a one-parameter subgroup? *Yes/No?*

Does any direct affine transformation of the plane belong to a one-parameter subgroup? *Yes/No?*

Examination ticket # 27.

1. *Give a complete proof of the following statement:* The Riemannian metric in the upper half-space model of hyperbolic space is preserved by inversions.

2. *Problem:* prove that a direct similarity of the plane, which is not an isometry, and a parallel translation never commute.

3. *Answer the following question* (*proofs are optional*)*:* List all finite groups of rotations in three–dimensional space.

Examination ticket # 28.

1. *Give a complete proof of the following statement:* The tangent space to a linear Lie group is a Lie algebra, i.e., it is invariant with respect to the bracket operation.

2. *Problem:* Find all isometries of the hyperbolic plane (direct and opposite) in the upper half–plane model which preserve the equidistant curve $\Im z = 2\Re z$.

3. *Answer the following question* (*proofs are optional*)*:* What is the minimal positive angle of rotation in a finite group G of rotations in $\mathbb{R}^3$ if it is known that G is neither cyclic, nor dihedral?

Examination ticket # 29.

1. *Give a rigorous account with complete proofs:* Describe the Lie algebras of $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SO}(n,\mathbb{R})$.

2. *Problem:* A direct isometry of T the hyperbolic plane preserves two different equidistant curves e_1 and e_2. Prove that either T^2 is the identity or e_1 and e_2 are equidistant form the same line.

3. *Answer the following questions* (*proofs are optional*)*:* Every similarity in $\mathbb{R}^3$ which is not an isometry has a fixed point. *Yes/No?*

8. Sources

Group theory:

G. Birkhoff, S. MacLane *A survey of modern Algebra*, Chapter 6.

A.G. Kurosh *Theory of groups*, vol. 1 Chapters 1,2

Geometry:

H.S.M. Coxeter *Introduction to geometry*, 1969, John Wiley.

Hyperbolic geometry:

S. Katok *Fuchsian groups*, Chapter 1, 1992, Chicago.