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ON THE WORK OF DOLGOPYAT ON PARTIAL AND NONUNIFORM HYPERBOLICITY

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ABSTRACT. This paper is a nontechnical survey and aims to illustrate Dolgopyat's profound contributions to smooth ergodic theory. I will discuss some of Dolgopyat's work on partial hyperbolicity and nonuniform hyperbolicity with emphasis on the interaction between the two—the class of dynamical systems with mixed hyperbolicity. On one hand, this includes uniformly partially hyperbolic diffeomorphisms with nonzero Lyapunov exponents in the center direction. The study of their ergodic properties has provided an alternative approach to the Pugh–Shub stable ergodicity theory for both conservative and dissipative systems. On the other hand, ideas of mixed hyperbolicity have been used in constructing volume-preserving diffeomorphisms with nonzero Lyapunov exponents on any manifold.

1. INTRODUCTION

Dmitry Dolgopyat, the winner of the second Brin Prize in Dynamical Systems, has made many fundamental contributions to various branches of the theory of dynamical systems. In this paper, I will describe some of Dolgopyat's results on partial hyperbolicity and nonuniform hyperbolicity, which range from constructing systems with nonzero exponents on compact smooth manifolds to studying accessibility of partially hyperbolic systems to constructing Sinai–Ruelle–Bowen (SRB) measures and effecting stable ergodicity for partially hyperbolic attractors. The common theme of the paper is a new emerging area in the theory of dynamical systems known as *mixed hyperbolicity* that is an interplay of uniform partial hyperbolicity and nonuniform complete hyperbolicity. In this paper, I will briefly describe the concept of mixed hyperbolicity and discuss some relevant results of Dolgopyat and other researches.

2. STABLE ERGODICITY

The concept of stable ergodicity in the context of smooth dynamics was introduced by Pugh and Shub (see for example, [10, 24]) and it is a great tool in studying genericity of ergodicity for smooth dynamical systems. Let $f: M \to M$ be a C^r diffeomorphism, $r \geq 1$, of a compact smooth connected Riemannian

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manifold M preserving a Borel probability measure μ. The map f is said to be *stably ergodic* if there exists a neighborhood $\mathscr{U}$ in the space $\mathrm{Diff}^k(M,\mu)$ of all C^k diffeomorphisms, $k \le r$, preserving the measure μ such that any C^r diffeomorphism $g \in \mathscr{U}$ is ergodic.[1] Similarly, one can define the notions of a system being *stably mixing*, *stably Kolmogorov*, *stably Bernoulli*, etc.

2.1. **Stable Ergodicity and Partial Hyperbolicity.** Anosov diffeomorphisms of compact smooth manifolds that preserve smooth measures (*i.e.,* measures that are equivalent to a volume) provide a simple example of stably ergodic (and indeed stably Bernoulli) maps. The next class of systems to consider is partially hyperbolic diffeomorphisms.[2] Recall that a diffeomorphism f is said to be *partially hyperbolic* if there is a df-invariant decomposition of the tangent bundle

$$TM = E^s \oplus E^c \oplus E^u$$

into *stable* E^s, *unstable* E^u, and *central* E^c subbundles, and if df uniformly expands and contracts along these subbundles with rates

$$\lambda_1 < \nu_1 \le \nu_2 < \lambda_2 \quad \text{and} \quad \lambda_1 < 1 < \lambda_2.$$

More precisely, for every $x \in M$ and $n \ge 0$, we have that

$$\|df(v)\| \le \lambda_1 \|v\| \quad \text{for } v \in E^s(x), \qquad \|df(v)\| \ge \lambda_2 \|v\| \quad \text{for } v \in E^u(x),$$
$$\nu_1 \|v\| \le \|df(v)\| \le \nu_2 \|v\| \quad \text{for } v \in E^c(x).$$

The notion of partial hyperbolicity was introduced in the early 1970s by Brin and Pesin [7] who were motivated by the study of frame flows. It also arose naturally from the work of Hirsch, Pugh and Shub [20] on normal hyperbolicity.

The distributions E^s and E^u are (Hölder) continuous in x and are uniquely integrable to invariant transverse continuous foliations W^s and W^u with smooth leaves.[3] These foliations are called *stable* and *unstable*, respectively, and they possess the *absolute continuity property*. The latter means that the conditional measures generated by volume m on local stable and unstable leaves are equivalent to the *leaf volumes* m^s and m^u (*i.e.,* the Riemannian volumes on leaves of the foliations generated by the Riemannian metric).

The central distribution E^c may or may not be integrable, and even if it is integrable, the central foliation W^c may not be absolutely continuous.

[1] In general, the number r may be strictly bigger than k; a typical case is $k = 1 < r = 2$.

[2] Pujals and Sambarino [24] have shown that if a diffeomorphism is stably ergodic, it has to be hyperbolic in some weak sense; more precisely, it must possess a *dominated splitting*.

[3] A partition W of the manifold M is said to be a *continuous foliation with smooth leaves* if there exist $\delta > 0$ and $\ell > 0$ such that for each $x \in M$, the following holds: 1) the element $W(x)$ of W containing x is a smooth ℓ-dimensional injectively immersed submanifold called the *global leaf* at x; the connected component $V(x)$ of the intersection $W(x) \cap B(x,\delta)$ that contains x is called the *local leaf* at x; 2) there exists a continuous map $\phi_x \colon B(x,\delta) \to C^1(D,M)$ (where $D \subset \mathbb{R}^\ell$ is the unit ball) such that for every $y \in M \cap B(x,\delta)$ the local leaf $V(y)$ is the image of the map $\phi_x(y) \colon D \to M$.

We now turn to the study of ergodicity of partially hyperbolic systems with respect to smooth invariant measures. A crucial role is played here by a property called *accessibility*. Let f be a partially hyperbolic diffeomorphism. We say that two points x and y are *accessible* if there is a path consisting of pieces of stable and unstable manifolds that connects these points. Clearly, accessibility is an equivalence relation; the equivalence classes are called the *accessibility classes*. Further, we say that f is *accessible* if any two points are accessible (*i.e.,* if there is only one accessibility class), and f is *essentially accessible* if the partition by accessibility classes is trivial (*i.e.,* any measurable set that consists of partition elements has either zero or full measure).

To illustrate the role of accessibility we consider the simple example of a volume-preserving partially hyperbolic map that is the direct product of an Anosov diffeomorphism of the torus T^n and the identity map of a compact manifold N. Note that for each $y \in N$, the set $T^n \times y$ is an ergodic component and at the same time is an accessibility class. Therefore, one can conjecture that *essential accessibility implies ergodicity*—the statement known as the *Pugh–Shub ergodicity conjecture*—for partially hyperbolic maps.[4] If this conjecture were true one could conclude that *stable essential accessibility implies stable ergodicity*.

At present, the conjecture has been proven under an additional technical assumption on the map known as *center-bunching*:

$$\lambda_1 < \nu_1 \nu_2^{-1} \quad \text{and} \quad \lambda_2 > \nu_2 \nu_1^{-1}.$$

Observe that center-bunching is an open property in the space of C^1 partially hyperbolic diffeomorphisms.

THEOREM 2.1 ([11])**.** *Let f be a C^2 partially hyperbolic diffeomorphism preserving a smooth measure μ. If f is essentially accessible and center-bunched, then it is ergodic. If in addition, f is stably essentially accessible, then it is stably ergodic in* $\mathrm{Diff}^1(M,\mu)$.

When the center direction is 1-dimensional, the center-bunching condition can be dropped, leading to a complete solution of the Pugh–Shub conjecture in this case.

THEOREM 2.2 ([11, 25])**.** *Let f be a C^2 partially hyperbolic diffeomorphism preserving a smooth measure μ. Assume that* $\dim E^c = 1$ *and that f is essentially accessible. Then f is ergodic. If in addition, f is stably essentially accessible, then it is stably ergodic in* $\mathrm{Diff}^1(M,\mu)$.

2.2. **Accessibility.** In view of the previous results, it is crucial to know if accessibility (and stable accessibility) is generic in some sense. The first and most general result in this direction was obtained by Dolgopyat and Wilkinson [18].

[4] The use of essential accessibility instead of accessibility is important as a partially hyperbolic ergodic automorphism of the torus is essentially accessible but is not accessible.

THEOREM 2.3. *Let $f \in \mathrm{Diff}^q(M)$ (respectively, $f \in \mathrm{Diff}^q(M,\mu)$), $q \ge 1$ be a partially hyperbolic diffeomorphism. Then for every neighborhood $U \subset \mathrm{Diff}^1(M)$ (respectively, $U \subset \mathrm{Diff}^1(M,\mu)$) of f there is a C^q diffeomorphism $g \in U$ that is stably accessible.*

It follows that stable accessibility is dense in the C^1 topology.

The proof of this result uses and substantially refines the quadrilateral argument introduced in [5]. It goes as follows (for simplicity we assume that the central bundle E^c is integrable to a foliation W^c). Given a point $p \in M$, consider a 4-legged path $[z_0, z_1, z_2, z_3, z_4]$ originating at $z_0 = p$. Connecting z_{i-1} with z_i by a geodesic γ_i lying in the corresponding stable or unstable manifold (in the induced Riemannian metric of these manifolds), we obtain the curve $\Gamma_p = \bigcup_{1\le i\le 4} \gamma_i$ and we parameterize it by $t \in [0,1]$ with $\Gamma_p(0) = p$. If the distribution $E^s \oplus E^u$ were integrable (and hence, the accessibility property would fail), then the endpoint $z_4 = \Gamma_p(1)$ would lie on the leaf of the corresponding foliation passing through p.

Therefore, one can hope to achieve accessibility by arranging a 4-legged path in such a way that $\Gamma_p(1) \in W^c(p)$ and $\Gamma_p(1) \ne p$. In this case, the path Γ_p can be homotoped through 4-legged paths originating at p to the trivial path in such a way that the endpoints stay in $W^c(p)$ during the homotopy and form a continuous curve. Such a situation is usually persistent under small perturbations of f and hence leads to stable accessibility.

If the center bundle is 1-dimensional, Theorem 2.3 can be strengthened: one can show that accessibility is an open dense property in the space of diffeomorphisms of class C^r, $r > 1$, [25].[5]

2.3. **Negative (positive) central exponents.** A natural way to relax the center-bunching condition is to consider its nonuniform version, that is, to carefully analyze the action of the diffeomorphism along its central direction and in particular, examine its Lyapunov exponents in this direction, *i.e.,* studying the case of mixed hyperbolicity. By doing so, it may be rewarding to consider the cases in which the Lyapunov exponents in the central direction are: (1) all negative or all positive; (2) all nonzero, *i.e.,* some negative and some positive; (3) all zero; (4) not all nonzero, *i.e.,* some zero. This approach was proposed by Burns, Dolgopyat and Pesin in [8] and to some extent was inspired by Dolgopyat's work [12, 16] where he obtained some quantitative information on the system in the nonuniformly hyperbolic and zero exponent cases.

More precisely, we say that a partially hyperbolic diffeomorphism f preserving a smooth measure μ has *negative (respectively, positive) central exponents* if there is a set $A \subset M$ of positive μ-measure such that for every $x \in A$ and every $v \in E^c(x)$ the Lyapunov exponent $\chi(x, v) < 0$ (respectively, $\chi(x, v) > 0$).[6]

[5]It shown in [19] that in the case of a 1-dimensional center bundle, accessibility is an open property in the C^2 topology.

[6]Clearly, we may assume without loss of generality that A is invariant.

THEOREM 2.4 ([8]). *Let f be a C^2 essentially accessible diffeomorphism preserving a smooth measure μ. Assume that f has negative (or positive) central exponents. Then f is ergodic.*

We outline the proof of this theorem. It is based on the simple yet crucial observation that since f has negative central exponents on the set A, it is nonuniformly hyperbolic on this set and methods of nonuniform hyperbolicity theory apply. In particular, f has at most countably many ergodic components of positive volume on A. On the other hand, since f is uniformly partially hyperbolic, the "size" of local leaves of the unstable foliation is uniformly bounded from below. This guarantees that every ergodic component of positive volume contains an open (mod 0) ball and hence is itself an open (mod 0) set. Hence, the set A is open (mod 0). One can now use essential accessibility and the fact that f preserves a smooth measure to conclude that almost every trajectory of f is dense in A. In particular, A has full measure and $f\restriction_A$ is topologically transitive. This implies that f is ergodic.

Surprisingly, under the same assumptions as in Theorem 2.4 one can show that f is stably ergodic.

THEOREM 2.5 ([8]). *Let f be a partially hyperbolic C^2 diffeomorphism that is essentially accessible and preserves a smooth measure μ. Assume that f has negative (or positive) central exponents. Then f is stably ergodic in* $\mathrm{Diff}^1(M,\mu)$.

We stress that, unlike Theorem 2.1, only essential accessibility is required to guarantee stable ergodicity of f and whether, under the condition of the theorem, f is actually stably essentially accessible is irrelevant (and not known).

The proof of this theorem goes as follows. Consider a diffeomorphism g that preserves μ and is δ-close to f in the C^1 topology. Since f is ergodic, there is $\alpha > 0$ such that the Lyapunov exponent $\chi(x,v) \le -\alpha$ for almost every $x \in M$ and $v \in E^c(x)$. It follows that

$$\int_M \ln\|df\restriction_{E^c_f(x)}\|\, d\mu(x) < -\alpha.$$

Since the central bundle E^c_g depends continuously on g in the C^1 topology, we can choose δ so small such that

$$\int_M \ln\|dg\restriction_{E^c_g(x)}\|\, d\mu(x) < -\alpha/2$$

and then conclude that $\chi_g(x,v) \le -\alpha/2$ for $v \in E^c_g(x)$ and x in a set A_g of positive measure. In other words, g has negative central exponents. Although g may not be essentially accessible, one can show that it is *ε-essentially accessible* (*i.e.,* every accessibility class enters every ε-ball) where $\varepsilon = \varepsilon(\delta) \to 0$ as $\delta \to 0$. This and the fact that g preserves a smooth measure implies that almost every trajectory of g is $\varepsilon/2$-dense (*i.e.,* it enters every $\varepsilon/2$-ball). Since the Lyapunov exponent of $g\restriction_{A_g}$ is uniformly away from zero, ideas from [1] about hyperbolic times can be used to obtain that every ergodic component of $g\restriction_{A_g}$ of positive measure contains a ball whose radius is at least ε. Repeating the above argument, one can show that the set A_g has full measure and that g is ergodic.

3. THE DISSIPATIVE CASE

In the previous section we discussed the conservative case, *i.e.,* diffeomorphisms that preserve smooth measures. We now turn to the dissipative case where one of the main problems is to construct "natural" invariant measures with "good" ergodic properties.

Let M be a compact smooth Riemannian manifold, $V \subset M$ an open set, and $f: V \to M$ a C^2 diffeomorphism of V onto its image. A compact invariant set $\Lambda \subset V$ is said to be an *attractor* if there exists an open neighborhood $U \subset V$ of Λ such that $\overline{f(U)} \subset U$ and $\Lambda = \bigcap_{n \geq 0} f^n(U)$. The set U is called the *topological basin of attraction.*

An attractor Λ is called *partially hyperbolic* if $f \restriction_\Lambda$ is partially hyperbolic, that is, the tangent bundle $T_\Lambda M$ admits an invariant splitting $T_\Lambda M = E^s \oplus E^c \oplus E^u$ into stable, center, and unstable subbundles, respectively.

The unstable distribution E^u is integrable to an unstable lamination W^u so the attractor Λ is the union of the global unstable manifolds of its points, *i.e.,* $W^u(x) \subset \Lambda$ for every $x \in \Lambda$.

An f-invariant measure μ on Λ is called a u*-measure* if for almost every $x \in \Lambda$ the conditional measure $\mu^u(x)$ generated by μ on the leaf $W^u(x)$ is equivalent to the leaf volume $m^u(x)$ on $W^u(x)$. In what follows, we will address the following problems related to u-measures:

1. existence of u-measures;
2. relations between u-measures and Sinai–Ruelle–Bowen (SRB) measures; in particular, between the basins of u-measures and the topological basin of attraction;
3. (non)uniqueness of u-measures;
4. u-measures with negative central exponents.

3.1. **Existence of u-measures.** Starting with the Riemannian volume m in a neighborhood U of Λ,[7] consider its evolution under the dynamics, *i.e.,* the sequence of measures

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} f_*^i m. \tag{3.1}$$

Any limit measure μ of this sequence of measures is concentrated on Λ.

THEOREM 3.1 ([23])**.** *Any limit measure μ is a u-measure.*

Fix $x \in \Lambda$ and consider a local unstable leaf $V^u(x)$ through x. We can view the leaf volume $m^u(x)$ on $V^u(x)$ as a measure on the whole of Λ. Consider its evolution, *i.e.,* the sequence of measures

$$\nu_n = \frac{1}{n} \sum_{i=0}^{n-1} f_*^i m^u(x). \tag{3.2}$$

Any limit measure ν of this sequence of measures is concentrated on Λ.

[7] One can also start with any measure that is absolutely continuous with respect to volume.

THEOREM 3.2 ([23])**.** *Any limit measure of the sequence* (3.2) *is a u-measure.*

For any ergodic u-measure ν and ν-almost every $x \in \Lambda$ the sequence of measures (3.2) converges to ν. Therefore, the class of all limit measures for sequences of type (3.2) coincides with the class of all u-measures, while the class of limit measures for sequences of type (3.1) may be smaller.[8]

3.2. **The basin of the measure.** Given an invariant measure μ on Λ, define its *basin* $B(\mu)$ as the set of points $x \in M$ for which the Birkhoff averages

$$S_n(\varphi)(x) = \frac{1}{n}\sum_{k=0}^{n-1}\varphi\big(f^k(x)\big)$$

converge to $\int_M \varphi\, d\mu$ as $n \to \infty$ for all continuous functions φ.

If Λ is a hyperbolic attractor, then μ is an SRB measure if and only if its basin has positive measure. If Λ is a hyperbolic attractor, then by a result in [3] *any measure with basin of positive volume is a u-measure.*

While any partially hyperbolic attractor has a u-measure, measures with basins of positive volume need not exist: just consider the product of the identity map and a diffeomorphism with a hyperbolic attractor. That is why the following result by Dolgopyat [13] is of great importance.

THEOREM 3.3. *If there is a unique u-measure for f in Λ, then its basin has full volume in the topological basin of Λ.*

In the case of a hyperbolic attractor, topological transitivity of $f\restriction_\Lambda$ guarantees that there is a unique u-measure for f on Λ (which is the unique SRB measure). In contrast, in the partially hyperbolic situation, even topological mixing may not guarantee that there is a unique u-measure. Indeed, consider $F = f_1 \times f_2$, where f_1 is a topologically transitive Anosov diffeomorphism and f_2 a diffeomorphism close to the identity. Then any measure $\mu = \mu_1 \times \mu_2$, where μ_1 is the unique SRB measure for f_1 and μ_2 any f_2-invariant measure, is a u-measure for F. Thus, F has a unique u-measure if and only if f_2 is uniquely ergodic. On the other hand, F is topologically mixing if and only if f_2 is topologically mixing.

3.3. ***u*-measures with negative central exponents.** We describe a special class of u-measures with "good" ergodic properties, which is an extension of the class of SRB measures for hyperbolic attractors to partially hyperbolic attractors. Consider a u-measure μ for f. We say that f has *negative central exponents* if there is a subset $A \subset \Lambda$ with $\mu(A) > 0$ such that the Lyapunov exponents $\chi(x, v) < 0$ for any $x \in A$ and $v \in E^c(x)$.[9]

The study of ergodic properties of u measures with negative central exponents was conducted by Bonatti and Viana in [4] and independently (in a somewhat different way) by Burns, Dolgopyat, Pesin and Pollicott in [9]. It addresses

[8]We note that in the case of (completely) hyperbolic attractors, the classes of limit measures for sequences of types (3.1) and (3.2) coincide.

[9]As before, without loss of generality we may assume that A is invariant.

the problems of uniqueness, ergodic properties of u-measures with negative central exponents and their relations to SRB measures. Let us point out that the role of the accessibility condition in the dissipative case is played by the requirement that every global unstable manifold is dense in the attractor.[10]

THEOREM 3.4 ([9]). *Let f be a C^2 diffeomorphism with a partially hyperbolic attractor Λ. Assume that*

1. *there exists a u-measure μ for f with negative central exponents;*
2. *for every $x \in \Lambda$ the global unstable manifold $W^u(x)$ is dense in Λ.*

Then the following statements hold:

1. *μ is the only u-measure for f and hence, the unique SRB measure;*
2. *f has negative central exponents at μ-almost every $x \in \Lambda$ and the map f is ergodic and is indeed Bernoulli;*
3. *the basin of μ has full volume in the topological basin of Λ.*

The proof of this theorem follows the line of arguments in the proof of Theorem 2.4, but is appropriately adapted to the dissipative case.

3.4. **Constructing measures with negative central exponents.** There are partially hyperbolic attractors for which *any* u-measure has zero central exponents (*e.g.,* the product of an Anosov map and the identity map of any manifold).

There are partially hyperbolic attractors that allow u-measures with negative central exponents, but not every global manifold $W^u(x)$ is dense in the attractor (*e.g.,* the product of an Anosov map and the map of the circle leaving north and south poles fixed).

Shub and Wilkinson [28] considered small perturbations F of the direct product map $F_0 = f \times \mathrm{Id}$, where f is a linear Anosov diffeomorphism of the 2-torus and the identity acts on the circle. They constructed F in such a way that it preserves volume, has negative central exponents on the whole of M, and its central foliation is not absolutely continuous.[11] Barraveira and Bonatti [2] obtained a multidimensional version of the above result by showing that if all the Lyapunov exponents in the central directions are zero, then, by an arbitrarily small perturbation, one can make their sum negative on a set of positive measure. Ruelle [26] extended the result of Shub and Wilkinson in another direction by showing that for an open set of one-parameter families of (not necessarily volume-preserving) maps F_ϵ through F_0, each map F_ϵ possesses a u-measure with negative central exponent.

Dolgopyat obtains a number of remarkable results on the existence of measures with negative central exponents in various situations. They are corollaries of his principal work [14] on stability of stochastic behavior. In this paper he considers a one-parameter family f_ϵ of C^∞ partially hyperbolic diffeomorphisms, where f_0 is an Anosov element in a standard abelian Anosov action

[10] In particular, density of every global unstable manifold implies that for any u-measure μ, almost every trajectory is dense in $\operatorname{supp}\mu$.

[11] This is an interesting "pathological" phenomenon known as *Fubini's nightmare*. One can show that this phenomenon is persistent under small perturbations of the map F.

with sufficiently strong mixing properties. If μ is a unique SRB measure for f_0 and μ_ϵ a u-measure for f_ϵ, Dolgopyat shows that for any C^∞ function φ, the map

$$\varphi \mapsto \int \varphi \, d\mu_\epsilon$$

is differentiable at $\epsilon = 0$ and he obtained a formula for the derivative of this map. This result is an extension to the case of partially hyperbolic systems of a similar result by Ruelle [27] for Anosov maps.[12] This result has many applications to studying some delicate stochastic properties of dynamical systems such as group extensions over Anosov maps and small perturbations of the time-1 map of Anosov flows. Other applications include:

1. One-parameter families f_ϵ of maps where f_0 is the time-1 map of the geodesic flow on the unit tangent bundle of a negatively curved surface. Dolgopyat proved that in the conservative case (*i.e.,* the maps f_ϵ are volume-preserving), generically either f_ϵ or f_ϵ^{-1} has negative central exponent for small ϵ and there is an open set of nonconservative families where the central exponent is negative for any u-measure.
2. Systems with zero central exponents subjected to rare kicks. Given diffeomorphisms f and g, let $F_n = f^n \circ g$. Dolgopyat proved that if f is either a T^1-extension of an Anosov diffeomorphism or the time-1 map of an Anosov flow and g is close to Id, then, for a typical g and any sufficiently large n, either F_n or F_n^{-1} has negative central exponent with respect to any u-measure.

In addition, let us mention another result of Dolgopyat [15], where he showed that in the class of skew products, negative central exponents appear for generic perturbations and that there is an open set of one-parameter families of skew products near $F_0 = f \times \text{Id}$ (f is an Anosov diffeomorphism and Id is the identity map of any manifold) where the central exponents are negative with respect to any u-measure.

4. STABLE ERGODICITY FOR DISSIPATIVE SYSTEMS

Let f be a C^2 diffeomorphism with a partially hyperbolic attractor Λ_f. Any C^1 diffeomorphism g that is sufficiently close to f in the C^1 topology, has a hyperbolic attractor Λ_g that lies in a small neighborhood of Λ_f. The stable ergodicity problem for partially hyperbolic attractors utilizes the notion of u-measures and can be stated as follows. We say that a C^r partially hyperbolic diffeomorphism f is *stably ergodic* if there is a neighborhood U of f in the C^k topology, $1 \le k \le r$, such that any C^r diffeomorphism $g \in U$ possesses a unique

[12] Dolgopyat's approach is substantially different from [27] and can be used to study differentiability for even more general classes of systems.

u-measure μ_g that is supported on the attractor Λ_g and g is ergodic with respect to μ_g.[13] Similarly, one can define the notion of *stably mixing*, *stably K* and *stably Bernoulli*.

The approach that utilizes measures with negative central exponents turns out to be quite successful (and at present is the only available) in establishing stable ergodicity for maps with partially hyperbolic attractors. It was developed in the work of Burns, Dolgopyat, Pesin and Pollicott [9].

THEOREM 4.1. *Let f be a C^2 diffeomorphism with a partially hyperbolic attractor Λ_f. Assume that*

1. *there is a u-measure μ for f with negative central exponents on a subset $A \subset \Lambda_f$ of positive measure;*
2. *for every $x \in \Lambda_f$ the global strongly unstable manifold $W^u(x)$ is dense in Λ_f.*

Then f is stably ergodic (indeed, stably Bernoulli). More precisely, for any C^2 diffeomorphism g that is sufficiently close to f in the $C^{1+\alpha}$-topology for some $\alpha > 0$, the following statements hold:

1. *g has negative central exponents on a set of positive measure with respect to a u-measure μ_g;*
2. *the measure μ_g is the unique u-measure (and hence the unique SRB measure) for g;*
3. *the map $g\restriction_{\Lambda_g}$ is ergodic with respect to μ_g (indeed is Bernoulli);*
4. *the basin $B(\mu_g)$ has full volume in the topological basin of Λ_g.*

We stress that (similar to the conservative case) the condition that every leaf of the unstable foliation is dense in the attractor is required only for the unperturbed map f and that the stable ergodicity result holds regardless whether the perturbation map g satisfies this condition or not.[14]

4.1. **Attractors with positive central exponents.** For partially hyperbolic systems preserving smooth measures, the case of u-measures with positive central exponents can be trivially reduced to the case of u-measures with negative central exponents by reversing the time. This is not true for dissipative partially hyperbolic systems and the study of u-measures with positive central exponents is more challenging. The first ergodicity result in this direction was obtained in [1] under the stronger assumption that there is a set of positive volume in a neighborhood of the attractor with positive central exponents.

Stable ergodicity of partially hyperbolic attractors with positive central exponents was studied in [29], where a result similar to Theorem 4.1 is proven.

[13] Of course, this implies that the unperturbed map f is ergodic with respect to its unique u-measure μ_f.

[14] One can show that for every $\varepsilon > 0$ there is a neighborhood of f in the C^1 topology such that every diffeomorphism g in this neighborhood has the property that every unstable global leaf is ε-dense in the attractor. This property along with the fact that $\chi(x, v) < -\alpha$ (for almost every $x \in \Lambda_g$, every $v \in E^c_g(x)$ and some $\alpha > 0$) is sufficient to establish ergodicity of g.

THEOREM 4.2. *Let f be a C^2 diffeomorphism with a partially hyperbolic attractor Λ_f. Assume that*

1. *there is a unique u-measure μ for f and μ has positive central exponents on a subset $A \subset \Lambda_f$ of* full *μ-measure;*
2. *for every $x \in \Lambda_f$, the global strongly unstable manifold $W^u(x)$ is dense in Λ_f.*

Then f is stably ergodic.

5. EXISTENCE OF NONUNIFORMLY HYPERBOLIC DYNAMICAL SYSTEMS ON ANY MANIFOLD

It has been a long-standing problem in hyperbolic dynamics to show that any compact smooth Riemannian manifold carries a volume-preserving Bernoulli diffeomorphism with nonzero Lyapunov exponents. In the 2-dimensional case, this problem was solved by Katok [22]. For any manifold of dimension greater than 4, Brin [6] later constructed a volume-preserving Bernoulli diffeomorphism whose Lyapunov exponents all but one are nonzero. The final solution—that is to remove the remaining zero exponent in Brin's example and to also solve the problem in the 3- and 4-dimensional cases—was obtained by Dolgopyat and Pesin [17] using some techniques in mixed hyperbolicity that we mentioned above.[15]

THEOREM 5.1. *Given a compact smooth Riemannian manifold $M \neq S^1$, there exists a C^∞ diffeomorphism f of M such that*

1. *f preserves the Riemannian volume m;*
2. *f has nonzero Lyapunov exponents almost everywhere;*
3. *f is a Bernoulli diffeomorphism.*

5.1. Katok's Example.

The main step in Katok's proof of this theorem in the 2-dimensional case is a construction of an area-preserving C^∞ Bernoulli diffeomorphism g of the unit disk D^2 in the plane that has the following properties:

- K_1: g has nonzero Lyapunov exponents almost everywhere.
- K_2: g has three fixed points p_1, p_2 and p_3, and is uniformly hyperbolic outside a small neighborhood U of the *singularity set* $S := \partial D^2 \cup \{p_1, p_2, p_3\}$, *i.e.,* there exists $\lambda < 1$, such that for every $x \notin U$,

$$\|dg\restriction_{E^s_g(x)}\| \le \lambda \quad \text{and} \quad \|dg^{-1}\restriction_{E^u_g(x)}\| \le \lambda.$$

- K_3: g has two invariant stable and unstable foliations, W^s_g and W^u_g, of $D^2 \setminus S$ with smooth leaves. These foliations are continuous and indeed are absolutely continuous. Furthermore, they are transverse everywhere except for points in the singularity set.
- K_4: $g\restriction_{\partial D^2} = \mathrm{Id}$ and the partial derivatives of $g(x)$ approach zero sufficiently fast as x approaches the boundary ∂D^2.

[15] The solution of a similar problem about existence of nonuniformly hyperbolic continuous-time dynamical systems on every compact smooth Riemannian manifold of dimension ≥ 3 is obtained in [21].

5.2. **Brin's Example.** Consider a compact smooth Riemannian manifold of dimension $n \geq 5$. Brin's construction consists of three steps.

Step 1. Starting from a volume-preserving hyperbolic automorphism A of the torus $\mathbb{T}^{n-3}$ consider the suspension flow $\tilde{T}^t$ over A with a constant roof function. This flow is Anosov but does not have the accessibility property. However, one can perturb the roof function in such a way that the new flow T^t (which is still Anosov) does have the accessibility property. The phase space Y^{n-2} of T^t is diffeomorphic to the product $\mathbb{T}^{n-3} \times [0,1]$, where the tori $\mathbb{T}^{n-3} \times \{0\}$ and $\mathbb{T}^{n-3} \times \{1\}$ are identified by the action of A.

Step 2. Consider the skew product map R on $K = D^2 \times Y^{n-2}$ given by

$$R(x, y) = \big(g(x), T^{\alpha(x)}(y)\big),$$

where α is a nonnegative function on D^2 that is equal to zero in a neighborhood U of the singularity set S and is strictly positive otherwise. Denote by $\Gamma = S \times Y^{n-2}$ the *singularity set* for R, and set $\Omega = (D^2 \setminus U) \times Y^{n-2}$. The map R has the following properties:

B_1: R is nonuniformly partially hyperbolic on $K \setminus \Gamma$, *i.e.,*

$$T_z K = E^s_R(z) \oplus E^c_R(z) \oplus E^u_R(z), \quad z \in K \setminus \Gamma.$$

B_2: R is uniformly partially hyperbolic on Ω, *i.e.,* for some $\mu < 1$ and every $z \in \Omega$,

$$\|dR\restriction_{E^s_R(z)}\| \leq \mu, \quad \|dR^{-1}\restriction_{E^u_R(z)}\| \leq \mu.$$

B_3: The distributions $E^s_R(z)$ and $E^u_R(z)$ generate two continuous foliations W^s_R and W^u_R on $K \setminus \Gamma$ with smooth leaves. These foliations are absolutely continuous. Furthermore, they are transverse everywhere except for the points in the singularity set.

B_4: R has the essential accessibility property with respect to the foliations W^s_R and W^u_R.

Step 3. There is a smooth embedding

$$\chi_1 \colon K = D^2 \times Y^{n-2} \to B^n,$$

that is a diffeomorphism except for the boundary ∂K (B^n is the unit ball in $\mathbb{R}^n$). There is a smooth embedding $\chi_2 \colon B^n \to M$ that is a diffeomorphism except for the boundary ∂B^n. Since the map R is the identity map on the boundary ∂K, the map

$$h = (\chi_1 \circ \chi_2) \circ R \circ (\chi_1 \circ \chi_2)^{-1} \colon M \to M$$

has the following properties:

1. h preserves the Riemannian volume;
2. h is a Bernoulli diffeomorphism;
3. h has only one zero Lyapunov exponent in the central direction for R.

We now outline the approach developed by Dolgopyat and Pesin [17] that allows one to remove the zero exponent in Brin's example by making a sufficiently small perturbation of the map R, creating negative Lyapunov exponent in this direction. To this end, one can show that given $r > 0$ and $\varepsilon > 0$, there is a C^r diffeomorphism $P\colon K \to K$ that preserves volume m and is such that

DP_1: $d_{C^r}(P,R) \le \varepsilon$ and P is *gentle, i.e., P* is concentrated outside the singularity set Γ meaning that $P(x) = R(x)$ for x outside a small neighborhood of Γ;

DP_2: almost every orbit of P is dense in K;

DP_3: for almost every $z \in K$ there exists a decomposition

$$T_zK = E^s_P(z) \oplus E^c_P(z) \oplus E^u_P(z)$$

such that $\dim E^c_P(z) = 1$ and

$$\int_K \chi^c_P(z)\, dm < 0,$$

where

$$\chi^c_P(z) = \lim_{n\to\infty} \frac{1}{n} \log\| df^n\restriction_{E^c_P(z)}\|$$

is the Lyapunov exponent at $z \in K$ in the central direction.

The desired map P can be constructed in the form $P = \varphi\circ R$ where the perturbation φ is given as follows. Fix a point $z_0 \in K\setminus\Gamma$ and choose a coordinate system $\{x,\xi\}$ in a small ball $B(z_0,r)$ around z_0 of radius $r > 0$[16] such that $z_0 = (x_0,\xi_0)$, $dm = \rho(x,\xi)\, dx\, d\xi$ and

$$E^c_R(y_0) = \frac{\partial}{\partial\xi_1}, \quad E^s_R(y_0) = \left\langle \frac{\partial}{\partial\xi_2},\dots,\frac{\partial}{\partial\xi_k}\right\rangle, \quad E^u_R(y_0) = \left\langle \frac{\partial}{\partial\xi_{k+1}},\dots,\frac{\partial}{\partial\xi_{n-2}}\right\rangle$$

for some k with $2 \le k < n-2$. Let $\psi(t)$ be a C^∞ function with compact support and let $\tau = \frac{1}{r^2}(\|x\|^2 + \|\xi\|^2)$. Define

$$\varphi(x,\xi) :=$$
$$\Big(x,\, \xi_1\cos(\varepsilon\psi(\tau)) + \xi_2\sin(\varepsilon\psi(\tau)), -\xi_1\sin(\varepsilon\psi(\tau)) + \xi_2\cos(\varepsilon\psi(\tau)),\, \xi_3,\dots,\xi_{n-2}\Big).$$

Now the map

$$h = (\chi_1\circ\chi_2)\circ P\circ(\chi_1\circ\chi_2)^{-1}\colon M \to M$$

has all the desired properties (see Step 3 in Brin's construction described above).

In the three- and four-dimensional cases we consider the manifold $D^2 \times T^\ell$, where $\ell = 1$ in the three-dimensional case and $\ell = 2$ in the four-dimensional case. Further, we define the skew product map R by

$$R(z) = R(x,y) = \big(g(x),\, R_{\alpha(x)}(y)\big), \qquad z = (x,y),$$

where $R_{\alpha(x)}$ is the translation by $\alpha(x)$, and α is a nonnegative C^∞ function that is equal to zero in a small neighborhood of the singularity set S and is strictly positive otherwise. The map R is nonuniformly partially hyperbolic; its central direction is one-dimensional in the case $n = 3$ and it is two-dimensional in the case $n = 4$. One can now use a modification of the above argument to construct a C^∞ volume-preserving Bernoulli perturbation P of R such that: (1) if $n = 3$, the central exponent for P is negative and thus P is the desired map; (2) if $n = 4$, the sum of the two central exponents for P is negative. This of course does not

[16]The radius r should be chosen small so that the ball does not intersect the singularity set Γ.

exclude the case that one of the central Lyapunov exponents is positive and thus requires further perturbation to ensure that each of the two exponents is negative; this is quite a challenging problem that requires some sophisticated techniques.

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