INTRODUCTION TO DYNAMICAL SYSTEMS

A.Katok

PROBLEM SET # 1

Due to Friday, August 30

1. Consider all natural numbers consisting of exactly 1000 digits. What proportion of these contain all digits ? Calculate the answer to the second decimal digit and provide a rigorous justification of the answer.

2. What is the probability to have exactly 500 heads in a sequence of 1000 coin tossings? Find an exact formula, calculate the answer to the second decimal digit and provide a rigorous justification of the answer.

3. Consider the dynamical system (the model from population biology, x_n represents the size of a population, a is the reproduction rate) $x_{n+1} = ax_n(1 - x_n)$, where $x_0 \in (0, 1)$. Suppose that the population dies out, that is $x_n \to 0$. What restriction it imposes on the value of a?

Now suppose the size of the population stabilizes, i.e. $x_n \to A > 0$. What restriction it imposes on the value of a?

4. Assume that it is known that for every pair of points on the sphere that aren't diametrically opposite, there is a unique shortest piecewise smooth curve which connects the points. Prove that the curve is an arc of a big circle.

5*. Prove the statement of the Exercise 4 without assuming existence or uniqueness of the shortest curve.

6. Let f_n be the n-th Fibonacci number. Prove using the Contraction Principle that

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n}$$

exists and calculate it.

7. Suppose that f is a map of a closed interval I into itself satisfying the following contition which is weaker than the assumption for the Contraction Princicle:

$$d(fx, fy) < d(x, y) \text{ for any } x \neq y.$$

Prove that f has a unique fixed point $x_0 \in I$ and for any $x \in I$, $\lim_{n\to\infty} f^n x = x_0$.

8. Show that the assertion of the previous exercise is not valid for maps of the whole line. More specifically, construct an example of a map $f : \mathbb{R} \to \mathbb{R}$ (i.e. a real-valued function of one real variable) such that $d(fx, fy) < d(x, y)$ for any $x \neq y$, f has no fixed points. and for some x, y $d(f^n x, f^n y)$ does not converge to zero as $n \to \infty$.

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PROBLEM SET # 2

Due to Monday, September 9

9. Show that for any $n \times n$ matrix $A = (a_{ij})_{i,j=1,\dots,n}$ one has

$$\|A\| \le \sqrt{\sum_{i,j=1}^{n} a_{ij}^2}.$$

10. Show that for any $n \times n$ matrix $A = (a_{ij})_{i,j=1,\dots,n}$ one has

$$\|A\| \ge (|\det A|)^{\frac{1}{n}}.$$

11. Show that for an invertible matrix A at least one of the two quanities $\|A\|$ and $\|A\|^{-1}$ is greater or equal than one.

12. Prove that the norm of a matrix is a continuous function of its coefficients.

13. Let $f : [0,1] \to [0,1]$ be a *non–increasing* continuous map. What are possible periods for periodic points for such a map?

14. Consider the quadratic familly $f_a :, \ f_a(x) = ax(1-x)$. Prove that for $a \ge 4$ for any natural number n the map f_a has 2^n periodic points of period n on the interval $[0,1]$, i.e. the iterate f^n has 2^n fixed points.
Hint: Use the Intermediate Value Theorem.

15.Prove that for $a \ge 4$ for any natural number n the map f_a has a periodic point of minimal period n on the interval $[0,1]$,

Note: *The last two problems provide a glimpse into a genuinely complicated behavior in a dynamical system. Studying and understanding such behavior will be one of our main tasks later in this course.*

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PROBLEM SET # 3

Due on Friday, September 13

"CONTROL PROBLEM"

16. Consider the quadratic map $f_a : \mathbb{R} \to \mathbb{R}$. Let $a > 4$ and assume that $x \in [0,1]$, $f_a(x) \notin [0,1]$. Prove that $f^n(x) \to -\infty$ as $n \to \infty$.

NOTE: *Please do not consult with other students while solving problem N.16. If you have any questions about it address those to the course teaching assistant.*

17. Consider the following map $f : \mathbb{R} \to \mathbb{R}$,

$$f(x,y) = (\frac{5x+3y}{4} - (x+y)^2, \frac{3x+5y}{4} - (x+y)^2).$$

a) Find all fixed points of the map.
b) Find which of these points are attracting.
c) Prove that the iterates $f^n(0.9, -0.1)$ converge and find the limit.
d) Prove that the iterates $f^n(0.1, -0.1)$ converge and find the limit.

18. Let $I \subset \mathbb{R}$ be an interval and $f : I \to I$ be a map. Assume that x_0 is a fixed point of f and $|f'(x_0)| > 1$. Prove that x_0 is a repelling fixed point not assuming that f' is continuous or even that it exists everywhere.

19. Consider the system of differential equations on the plane, i.e. for $x = (x_1, x_2)$, we have

$$\frac{dx}{dt} = F(x) = (F_1(x), F_2(x)).$$

Assume that $\frac{d(x_1^2+x_2^2)}{dt} < 0$ for $x \neq 0$. Prove that for any initial condition solutions of this system converge to 0 as $t \to \infty$.

OPTIONAL PROBLEM

20. Under the assumptions of the previous problem let $S_t,\ t \in \mathbb{R}$ be the flow defined by the solutions of these equations.
   a) Prove that is F is a linear vector function then S_1 is a contracting map.
   b) Is this conclusion always true without the linearity assumption?

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PROBLEM SET # 3A

Assignments 3 and 3A are due on Friday, September 20

"CONTROL PROBLEM"

22. Let f be a map of an interval with continuous derivative and with a fixed point x_0 such that $f'(x_0) \neq 1$. Prove that there exists a neighborhood U of the fixed point x_0 with the following property: there exists an $\epsilon > 0$ such that any differentiable map g with $|f - g| < \epsilon$ and $|f' - g'| < \epsilon$ has exactly one fixed point in U.

Please do not consult with other students while solving problem N.22. If you have any questions about this problem address those to the course teaching assistant.

23. Consider a *linear* map of the plane $\mathbb{R}^2$. We will ignore the origin which is always a fixed point. Prove that one of the following possibilites hold:
 (i) there are no other periodic points;
 (ii) all points are periodic with the same minimal period;
 (iii) all points are periodic with the minimal period either one or two;
 (iv) some points have period one and the rest are not periodic;
 (v) some points have period two and the rest are not periodic.

24. Prove that a linear map in the three–dimensional space $\mathbb{R}^3$ cannot have simultaneously periodic orbits with minimal period three and four.

25. Consider the following family of maps for the values of parameter τ near zero of the line which exhibits a bifurcation of the fixed point at zero from an isolated repeller to an attracting point with two repelling points on its sides:

$$F_\tau(x) = x - \tau x + x^3.$$

Show that this family can be perturbed by adding an arbitrary small smooth term into a family which has for all values of the parameter an isolated repelling fixed point near zero and at a certain value of the parameter a simplest bifurcation takes place near zero, a pair of fixed points, one attracting and out repelling, appears out of nothing.

OPTIONAL PROBLEMS; no deadline.

26*.(*Period three implies chaos*). Let f be a continuous map of a closed interval into itself. Suppose f has a non–fixed periodic point of period three. Prove that for any natural number n there is a periodic point for f whose minimal period is equal to n.

27*. Suppose a continuous map f of a closed interval I into itself has only one periodic orbit. Prove that the orbit is an attracting fixed point x_0,i.e. for every point $x \in I,\ f^n x \to x_0$ as $n \to \infty$.

28**. Prove that in the standard quadratic family f_a periodic points with arbitrary minimal periods exist for $a \geq \sqrt{8}$.

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PROBLEM SET # 4

Due to Friday, September 27

"CONTROL PROBLEMS"

29.Consider a norm in $\mathbb{R}^n$ with the following extra property (monotonicity)
If $0 \le x_i \le y_i,\ i = 1, \dots, n$ then $\|(x_1, \dots, x_n)\| \le \|(y_1, \dots, y_n)\|$.
Given metric spaces $X_1, \dots, X_n$, prove that the formula

$$(*) \qquad d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \|(d_{X_1}(x_1, y_1). \dots, d_{X_n}(x_n, y_n))\|$$

defines a metric in the product space $X_1 \times \dots \times X_n$ and show that all such metrics (defined by different norms) are Lipschitz equivalent.

30. We will say that a norm in $\mathbb{R}^n$ is *of Euclidean type* if it is equal to the square root of quadratic form of coordinates:

$$\|(x_1, \dots, x_n)\|^2 = \sum_{i \le i \le j \le n} a_{ij} x_i x_j.$$

Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map eventually contracting with respect to some (and hence any) norm. Prove that there exists a norm of Euclidean type for which L is a contracting map.

Please do not consult with other students while solving probles NN.29,30. If you have any questions about this problem address those to the course teaching assistant.

31. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map which has two different pairs of complex conjugate eigenvalues of absolute value one. Prove that there is a subset $T \subset \mathbb{R}^n$ invariant under L which in properly chosen coordinates looks as the two–dimensional torus.

32. In the setup of Problem 29 drop the monotonicity condition for the norm. Give an example when formula (*) does not define a metric.

33. For a given natural number $n \ge 2$ let $A_N = \{0, 1, \dots, N-1\}$ and let Ω_N be the space of all infinite sequences of elements from the "alphbet" A_N. Define the distance in Ω_N as follows: for $\omega = (\omega_1, \omega_2, \dots)$ and $\omega' = (\omega'_1, \omega'_2, \dots)$:

$$\mathrm{dist}(\omega, \omega') = \sum_{n=1}^{\infty} \frac{|\omega_n - \omega'_n|}{2^n}.$$

Prove that this makes Ω_N into a metric space homeomorphic to the Cantor set.

34. Consider the set of real numbers from the interval [0,1] which have a decimal representation in which any two adjacent digits are different. Prove that this set is homeomorphic to the Cantor set.

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PROBLEM SET # 5

Due on Friday, October 4

"CONTROL PROBLEMS"

35.The standard (middle-third) Cantor set C is identified with the space Ω_2 of infinite sequences of zeroes ans ones in two ways:
(i) By replacing twos by ones in the representation in base three;
(ii) By coding the points of C according to the way their images under iterates of the map f:

$$f(x) = 3x \text{ if } x \leq 1/2,\ \ 3 - 3x \text{ if } x \geq 1/2$$

visit intervals $\Delta_0 = [1, 1/3]$ and $\Delta_1 = [2/3, 1]$, Denote the corresponding maps $C \to \Omega_2$ by H_1 and H_2. Calculate the map $H_1 \circ H_2^{-1}$.

36.Consider an arc $\Delta \subset S^1$. For a rotation R_α define the *first return map* $F : \Delta \to \Delta$ as follows: $F(x) = R_\alpha^{n(x)}$, where $n(x)$ is the minimal positive number n such that $R_\alpha^n \in \Delta$. Prove that F looks like that: there are two points $a, b \in \Delta$ which divide Δ into three arcs: $\Delta_1,\ \Delta_2,\ \Delta_3$ in that order. Then F moves each arc without changing its length, so that the order is reversed.

REGULAR PROBLEMS

37.Consider two metrics in the space Ω_2; the metric d_2 from problem 33 and the Hamming metric $d_H(\omega, \omega') = 2^{-\min\{n:\ \omega_n \neq \omega'_n\}}$. Prove that these metrics are Lipschitz equivalent.

38.Prove that for any $\omega,\ \omega' \in \Omega_2$ there exists a map preserving the Hamming metric which maps ω into ω'
Hint: Use group structures on Ω_2.

39.Consider the space Ω_2 with the structure of the group of *dyadic intgers*, i.e. the addition of infinite to the left binary "fractions", where addition is performed by carrying over to the left. Prove that the map $T_1 : \omega \to \omega + 1$ preserves Hamming metric and has a dense orbit.

40. Prove that for any irrational number α there exist infinitely many fractions p/q, with p and q relatively prime such that

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2}.$$

OPTIONAL PROBLEMS (no deadline)

41.Prove that for any point $\omega \in \Omega_2$ there exists a map $f_\omega : \ \Omega_2 \to \Omega_2$, contracting with respect to the Hamming metric and such that $f_\omega \omega = \omega$.

42.Prove that the map T_1 from problem 39 is minimal, i.e all of its orbits are dense.

43*. Let $0 < a < b < 1$ and let $I_1 = [0, a], \ I_2 = [a, b], \ I_3 = [b, 1]$. Consider the following map $f_{a,b} : \ [0, 1) \to [0, 1)$:

$$f_{a,b}(x) = x + 1 - a \text{ for } x \in I_1, \ x + 1 - a - b \text{ for } x \in I_2, \ x - 1 + b \text{ for } x \in I_3.$$

Find necessary and sufficient conditions for the map $f_{a,b}$ to have a dense orbit *Hint:* Use Problem 36.

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PROBLEM SET # 6

Due on Friday, October 11

"CONTROL PROBLEMS"

44. Prove that any orbit of a *rational* rotation of the circle consists of all vertices of a certain convex regular polygon inscribed into the circle.
Warning: The orbit may not follow the sides of the polygon.

45. In the sequence of first digits of powers of a certain natural number written in base eight at least four different digits are present. Prove that then all seven digits are present and each of them appears infinitely many times.

REGULAR PROBLEMS

46. Prove that for any natural numbers n and m there exists a differentiable monotone map of the circle which has exactly m distict periodic orbits of period n and no more periodic orbits.
Hint: Use your understanding of rational rotations and monotone maps of the interval.

47. Find the infimum of the numbers c satisfying the following property: there exists infinitely many rational numbers p/q with p and q realtively prime such that

$$|\sqrt{2} - \frac{p}{q}| < \frac{c}{q^2}.$$

Hint: Recall the proof of Liouville Theorem.

48. Show that the uniform distribution for irrational rotations does not in general hold for *countable* unions of intervals. In other words, consturct an irrational number α and a set U, the union of countably many disjoint intervals, the sum of whose lengths is equal to l, such that for some point $x \in S^1$,the average frequencies of visits to $U, \frac{F_U(x,n)}{n}$ with respect to the rotation R_α do not converge to l as $n \to \infty$.

49. Let Γ be an ellipse in the plane which bounds the domain of area A. Fix a number $a < A$ and consider the map $f_a : \Gamma \to \Gamma$ defined by the following geometric condition: the area bounded by the arc between x and $f_a(x)$ in the counterclockwise (positive) direction and the chord connecting these points is equal to a. Prove that for any given a either all orbits are periodic or the map is minimal (all orbits are dense). Prove that each of the two possibilities takes place for infititely many values of a.

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PROBLEM SET # 7

Due on Friday, October 18

"CONTROL PROBLEMS"

50. Prove that the maps $f(x) = x/2$ and $g(x) = x/3$ of the unit interval into itself are topologically congugate.

51. Let α be an irrational number. Prove that the set of values of the function $\sin t + \sin \alpha t,\ t \in \mathbb{R}$ is the open interval $(-2, 2)$.

REGULAR PROBLEMS

52.Prove that any dense sequence of points on the circle (of the unit interval) can be re–ordered to become uniformly distributed.

53. Prove that for any monotone continuous from the left function d on the unit interval such that $d(0) = 0,\ d(1) = 1$ there exists a sequence x_n asymptotically distributed according to this function.

54. Prove that an invertible linear mar of the plane is topologically congudate to a contracting map if and only if it is eventually contracting, i.e. equivalently both eigenvalues are different from zero and are strictly less than one in absolute value.

55. Consider the map described in Problem 49 with Γ the unit square,so that $A = 1$. Prove that the map $f_{1/8}$ has both periodic and non–periodic orbits.

56.Under the assumptions of Problem 53 assume that d is continuous and strictly monotone and view the unit interval as the circle. Prove that there exist a continuos invertible map (a homeomorphism) of the circle onto itself whose orbits have asymptotic distribution defined by the function d.
Hint: Use the concept of topological conjugacy.

OPTINAL PROBLEM (no deadline)

57. (Unifrom distribution on the line) Let α be an irrational number. Fix a natural number N and an interval Δ on the *real line*. Let $F_\Delta(N)$ be the number of the the numbers of the form $\{m + \alpha n, 0 \le m, n \le N - 1\{$ Prove that

$$\lim_{N\to\infty} \frac{F_\Delta(N)}{N} = length(\Delta).$$

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PROBLEM SET # 9

Due on Friday, November 8

62. Let α and β be two real numbers rationally independent with 1 . Prove that the sequence $a_n = \{n\alpha\} + \{n\beta\}$ where $\{x\}$ is the fractional part of x has an asymptotic distribution and find it.

63. Consider a round chamber which serves as a particle accelerator. The chamber is filled with a weakly radioactive material which glows under an impact of a very high concetration of particles. A very narrow parallel beam of particles issues from a very small source in the wall of the camera and photographs of the camera are taken form its top. What will the photographs show?
Note: You should interpret the non–rigorous terms such as "very high", "very narrow", "very small" in a reasonable way so that you can obtain an idealization producing a meaningful answer that could be rigorously justified.

64. Suppose a linear flow T^t_ω on the torus $\mathbb{T}^m$ contains a transformation $T_{t_0\omega}$ which is periodic but not equal to the identity. Prove that the closures of the orbits of the flow are circles.

65. Prove that for the middle–third Cantor set C, $E_3C = C$ and $E_{-3}C = C$, where $E_m x = mx (\mod 1)$

66. Consider points $x \in S^1$ such that for all $n = 0, 1, 2 \dots,\ E_2^n x$ does not belong to the open interval $(1/4, 1/2)$. Prove that such points form a Cantor–like set (closed, no isolated points, does not contain any interval)
Hint: Compare this with the previous problem.

67. Prove that the linear expanding maps $E_m,\ |m| \geq 2$ are pairwise not topologically congugate.

OPTIONAL PROBLEMS (no deadline)

68. Prove that the quadratic map f_4 is topologically conjugate to the "tent map": $t(x) = 1 - |2x - 1|$.

69. For any transformation $T_{t_0\omega}$ from the linear flow T^t_ω on the torus $\mathbb{T}^m$ the closures of its orbits either coincide with the closures of the orbits of the flow (which are tori of a certain dimension $k,\ 1 \leq k \leq m$) or are finite unions of tori of dimension $k - 1$.

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TOPICS FOR THE FINAL EXAMINATION:

Definitions and concepts:

Derivative (differential) of a map in a Euclidean space.
Norm of a matrix.
Metric space.
Complete metric space
Compact metric space.
Open and closed sets.
Homeomorphism between metric spaces.
Lipshitz equivalence of metric spaces.
Isometry between metric spaces.
Eventually contracting map.
Torus
Cantor set
Topological transitivity.
Topological mixing.
Topological conjugacy of dynamical systems.
Semi–conjugacy of dynamical systems.
Degree of a continuous map of the circle.
Rotation number of a circle homeomorphism.
Continuous–time dynamical systems (one–parameter groups of homeomorphisms).
Examples of linear differential eqations in the plane as continuous–time dynamical systems: node, saddle and focus
One- and two-sided sequence spaces.
Shifts in the sequence spaces.

RESULTS.

1. Contraction mapping principle for maps of a complete metric space.

2. Theorem: If a continuously differntiable map has a fixed point with the norm of the derivative less than one, than the map is contracting in a certain ball around the point.

3. Theorem: positive and negative iterates of any point under a monotone continuous maps of an interval converge to a limit and such limits are fixed points. points.

4. Structure of the quadratic maps for parameter values $0 \leq a \leq 3$. Three cases :$0 \leq a \leq 1$,$1 \leq a \leq 2$,$2 \leq a \leq 3$.(A ticket may ask for a proof in one of the cases and the statements for the others.)

5. Linear maps in dimension two as dynamical systems. Describe various types of behavior corresponding to various possibilituies for the eigenvalues. (A ticket may ask for a proof for a specific configuration of eigenvalues and statements in some other cases.)

6. Structure of the quadratic maps for large parameter values (greater than $2(1+\sqrt{2})$). Invariant Cantor set. Every point not in the set converges to infinity.

7. Theorem. If $k \neq 10^l$ where l is a natural number then there exists a power of k whose decimal representation begins with any given finite combination of digits.

8. Asymptotic distribution of the first digits of $k^n,\ k \neq 10^l$. Deduce from the uniform distribution for irrational rotations.

9. Invariant circles for a linear map in $\mathbb{R}^n$, which has a pair of non–real complex–conjugate eigenvalues.

10. Existence of rotation number for an orientation–preserving homeomorphism of the circle.

11. Reduction of the motion of two particles of equal mass on an interval to a billiard motion in a triangle and of the latter to a linear flow on the torus. Admissible angles; partial and complete unfolding.

12.Necessary and sufficient condition of uniform distribution for a translation $T_{(\alpha,beta)}$ on the two–torus: $1, \alpha$ and β are rationally independent).You have to be able to give a complete account of one of the two proofs given in class. The choice is yours. (A ticket may ask for a detailed proof of some steps and the statement of others)

13. Linear expanding maps E_m of the circle. Calculation of the number of periodic orbits. Existence of a periodic orbit with any given minimal period. Density of periodic orbits. Invariance of the standard Cantor set for E_3.

14. Criterion of topological transitivity for a continuous map f of a complete separable metric space (look at the latest version of notes). Corollary: Existence of a dense orbit for $E_m,\ |m| \geq 2$

15. Shifts in the sequence spaces (one–sided and two–sided). Calculation of the number of periodic points. Existence of a periodic orbit with any given minimal period. Density of periodic orbits. Existence of a dense orbit.

16. Tolological conjugacy between any two expanding maps of a given degree.

17. Hyperbolic automorphisms of the two–torus. Calculation of the number of periodic points. Density of periodic orbits. Topological mixing.

18. Construction of the “horseshoe” map. Topological conjugacy with the shift in the space of two–sided sequences.