FALL 2000. MATH 527: TOPOLOGY/GEOMETRY

A.Katok

Problem set 1; August 24, 2000

Due on Tuesday, September 5

1. Find all different topologies (up to a homeomorphism) on the sets consisting of 2 and 3 elements.

2. We define *Zariski Topology* in $\mathbb{R}^2$ by declaring zero sets of polynomials in two variables to be closed.
   a) Prove that this defines a topology.
   b) Show that it is (T1) but not (T2) (Hausdorff)

3. Consider the product topology on the product of countably many copies of the real line. (this product space is sometimes denoted $\mathbb{R}^\infty$).
   a) Does it have a countable base?
   b) Is it separable?

4. Consider the space $\mathcal{L}$ of all bounded maps $\mathbb{Z} \to \mathbb{Z}$ with the topology of pointwise convergece.
   a) Describe open sets for this topology.
   b) Prove that $\mathcal{L}$ is a ountable union of disjoint closed subsets each homeomorphic to a Cantor set.
   Hint: Consider the fact that the countable product of two–point spaces with the product topology is homeomorphic to a Cantor set.

5. Consider the *profinite* topology on $\mathbb{Z}$ where open sets are defined as unions (not necessarily finite) of ariphmetic progressions.
   a) Prove that this defines a topology.
   b) Show that it is Hausdorff but not discrete.

6. Consider the one–parameter group of homeomorphisms of the real line generated by the map $x \to 2x$. Consider three separation properties: (T2), (T1), and
   (T0) For any two points there exists an open set which contains one of them but not the other (but the one is not given in advance).
   Which of these properties does the factor–topology possess?

7. Prove that $\mathbb{R}$ (the real line) and $\mathbb{R}^2$ (the plane with the standard topology) are not homeomorphic. *Hint:* Use the notion of a connected set.

8. Prove the interior of any convex polygon in $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$.

ADDITIONAL PROBLEMS; submit solutions by September 19

A1. A topological space $(X, \mathcal{T})$ is called *regular* (or $(T3)$- space) if for any closed set $F \subset X$ and any point $x \in X \setminus F$ there exist disjoint open sets U and V such that $F \subset U$ and $x \in V$. Give an example of a Hausdorff topological space which is not regular.

A2. Proce that any open convex subset of $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$.

A3. A point x in a topological space is called *isolated* if the one-point set $\{x\}$ is open. Prove that any compact separable Hausdorff space without isolated points contains a closed subset homeomorphic to the Cantor set.

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Problem set 2 ; Settember 6, 2000

Due on Tuesday September 19

9. Find all different topologies (up to a homeomorphism) on a set consisting of 4 elements which make it a connected topological space.

10. The *profinite topology* on the group $\mathbb{Z}$ of integers is the weakest topology in which any arithmetic progression is an open set. Let $\mathbb{T}^\infty$ be the product of countably many copies of the circle with the product topology. Define the map $\varphi : \mathbb{Z} \to \mathbb{T}^\infty$ by

$$\varphi(n) = (\exp(2\pi in/2), \exp(2\pi in/3), \exp(2\pi in/4), \exp(2\pi in/5), \dots)$$

Show that the map φ is injective and that the topology induced on $\varphi(\mathbb{Z})$ coincides with profinite topology.

11. For a compact metric space X denote by $\mathcal{F}(X)$ the space of all closed subsets of X with the *Hausdorff metric* d_H:

$$d_H(A, B) = \max\{\max_{x\in A} \min_{y\in B} d(x, y), \max_{x\in B} \min_{y\in A} d(x, y)\}.$$

a) Prove that d_H is a metric.
b) Prove that the topology in $\mathcal{F}(X)$ induced by the Hausdorff metric does not depend on the metric in X defining the given topology.

12.A *topological group* is a group G endowed with a topology such that the group multiplication and taking inverse are continuous operations, i.e. the maps $G \times G \to G : (g_1, g_2) \to g_1 g_2$ and $G \to G : g \to g^{-1}$ are continuous.

Consider the group $SL(2, \mathbb{R})$ of all 2×2 matrices with determinant one with the topolology induced from the coordinate embedding into $\mathbb{R}^4$. Prove that it is a topological group.

13. Let X be a compact Hausdorff space. Prove that the space of continuous maps form X to the unit interval with the uniform metric is compact if and only if X contains finitely many elements.

14. Let $A \subset \mathbb{R}^2$ be the set of all vectors (x, y) such that $x + y$ is a rational number and $x - y$ is an irrational number.

Show that $\mathbb{R}^2 \setminus A$ is path connected.

15. A map $f : X \to Y$ between topological spaces is called a *homeomorphic embedding* if it is a homeomorphism between X and $f(X)$.

Construct a homeomorpghic embedding of $\mathbb{T}^\infty$ into the Hilbert cube.

16. For a metric space (X, d) and $r > 0$ let $B_d(r)$ be the minimum number of r-balls which cover X. Define the *upper box dimension* of X as

$$\limsup_{\epsilon \to 0} -\frac{\log B_d(\epsilon)}{\log \epsilon}.$$

Prove that spaces which are bi-Lipschtz equivalent have the same upper box dimension.

ADDITIONAL OPTIONAL PROBLEMS: Submit solutions by October 3

A4. Give an example of a compact metrizable path-connected topological space X such that no point of X has a connected neighborhood.

A5. Show that the closure of $\varphi(\mathbb{Z})$ as in problem 10 is homeomoprhic to the Cantor set. Introduce a translation-invariant metric on $\mathbb{Z}$ which generates the profinite topology and such that Cauchy sequences in that metric are exactly the sequences whose images under φ converge in $\mathbb{T}^\infty$.

A6. Consider the following subgroup S of $\mathbb{T}^\infty$, $S = \{(z_1, z_2, z_3, \dots) : z_n^2 = z_{n-1}, n = 2, 3, \dots\}$ with the topology induced from $\mathbb{T}^\infty$. Prove that as topological space S is connected but not path-connected.

A7. Consider the weakest topology in the set $\mathbb{R}$ or real numbers such that for any $t \in \mathbb{R}$ the function $x \to \exp(itx)$ is continuous. Prove that this topology is not metrizable.

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Problem set 3; September 19

Due on Tuesday October 3

17. A metric space X is called *precompact* if for any $\epsilon > 0$ it can be covered by finitely many ϵ-balls.

Prove that the completion of a metric space X is compact if and only if X is pre-compact.

18.Let $a_n \to 0$ be a sequence of positive numbers. A real number α is called $\{a_n\}$-*Diophatnine* is there exists a positive number C such that for any integers $p, \; q \neq 0$

$$|\alpha - \frac{p}{q}| > Ca_q.$$

Otherwise α is called $\{a_n\}$-*Liouvillean.*

Show that the set of all $\{a_n\}$-Diophantine numbers has first Baire category.

19.* Prove that the figure eight (i.e.the union of two circles with one common point) is not contractible.

20.* Prove that the product of a finite or countable collection of contractible spaces is contractible.

21. Construct a continuous map from the unit interval onto the Hilbert cube (Infinite-dimensional Peano curve).

22. Prove that the 2–torus with one point removed is not homeomorphic to any open set in the plane. *Hint:* Use Jordan curve theorem

23. Prove that no topological 3–manifold is homeomorphic to a topological 2–manifold.

24. Show that the standard Peano curve is a 1/2 Hölder map but does not satisfy α- Hölder condition with any $\alpha > 1/2$.

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*)Refers to material to be discussed on September 26.

ADDITIONAL OPTIONAL PROBLEMS; submit solutions by October 17

A8. A metric space X is called *locally path connected* if for any $\epsilon > 0$ there exists $\delta > 0$ such that any two points at a distance less than δ can be connected by a path contained in a ball of radius ϵ.

Prove that for any compact path connected and locally path connected subset X of the plane $\mathbb{R}^2$ there exists a continuous map $f : [0,1] \to \mathbb{R}^2$ whose image coincides with X (Generalized Peano curve).

A9. Consider the following subgroup S of $\mathbb{T}^\infty$, the product of countably many copies of the circle: $S = \{(z_1, z_2, z_3, \dots) : z_n^2 = z_{n-1}, n = 2, 3, \dots\}$ with the topology induced from $\mathbb{T}^\infty$. Prove that as a topological space S is connected but not locally path-connected at any point.

A10. Prove that in the space $C([0,1])$ of continuous functions on the unit interval the set of functions which are monotone on some interval has first category.

A11. Prove that $\varphi(\mathbb{Z})$ as in problem 10 is a subgroup of $\mathbb{T}^\infty$ which is isomorphic to the direct product of groups $\mathbb{Z}_p$ of of p-adic integers for all prime numbers $p = 2, 3, 5 \dots$.

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Problem set 4; October 10, 2000

Due on Tuesday, October 24

25. Any compact one–dimensional manifold is homeomorphic to the circle.

26. Write a complete argument for the statement: $\pi_1(S^1) = \mathbb{Z}$

27. For any finite cyclic group C there exists a compact connected three-dimensional manifold whose fundamental group is isomorphic to C.
Hint: Use Hopf fibration.

28. Complex projective plane $\mathbb{C}P(2)$ (which is four–dimensional manifold) is simply connected, i.e. its fundamental group is trivial.

29. Consider the following map f of the torus $\mathbb{T}^2$ into itself:

$$f(x, y) = (x + \sin 2\pi y, 2y + x + 2\cos 2\pi x) \pmod 1.$$

Describe the induced homomorphism f_* of the fundamental group.
Note: You may use the description of the fundamental group of the direct product $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

30. Let $X = \mathbb{R}^2 \setminus \mathbb{Q}^2$. Prove that $\pi_1(X)$ is uncountable.

31. A *tree* is a connected factor space of a finite or countable union of disjoint intervals with some endpoints indetified and no cycles.
Prove that any tree is contractible.

32. The real projective space $\mathbb{R}P(n)$ is not simply connected.
Note: You cannot simply refer to the general theory of covering spaces, but can use a covering argument with a lift to the sphere.

ADDITIONAL OPTIONAL PROBLEMS; submit solutions by November 7

A12. Prove that if a map $f : [0, 1] \to \mathbb{R}^2$ is such that the image of f contains an open ball, then f is not α Hölder for any $\alpha > 1/2$.

A13. For any abelian finitely generated group A there exists a compact manifold whose fundamental group is isomorphic to A.

A14. The fundamental group of any compact connected manifold is no more than countable and is finitely generated.

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MID-TERM EXAMINATION

Tuesday, October 10,2000

Do two problems from each section.

SECTION 1

1.1. Show that there exists a metric on the Cantor set such that there are only countably many different open balls .

1.2. Consider the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and let S be the factorspace obtained by identifying orbits of the map $I : x \mapsto -x$. Prove that S is homeomorphic to the sphere S^2.

1.3. Prove that the set of all real numbers x such that any represerntation of x in any base m contains infinitely many times each of the digits $\{0, 1, \ldots, m-1\}$ is the set of second Baire category.

1.4. Consider the space of all polynomials in one real variable equipped with a metric coming from a *norm*, i.e. $d(f, g) = \|f - g\|$, where
$\|f\| \geq 0$ and $\|f\| = 0$ implies that $f = 0$,
$\|\lambda f\| = |\lambda|\|f\|$ for $\lambda \in \mathbb{R}$,
$\|f + g\| \leq \|f\| + \|g\|$.
Prove that this space is not complete.

SECTION 2

2.1. Let X be the factorspace of the disjoint union of S^1 and S^2 with a pair of points $x \in S^1$ and $y \in S^2$ identified. Calculate $\pi_1(X)$.

2.2. The *binary tree* is the factor space of the countable disjoint union of closed intervals some of whose endpoints are identified in such a way that:
(i) the space is connected,
(ii) every endpoint of each interval is identified with exactly two other endpoints, and
(iii) there are no cycles, i.e one cannot follow a sequence of intervals without repetitions via identifications and come back.
Prove that the binary tree is contractible.

2.3. Torus with three points removed is homotopically equivalent to the bouquet of four circles.

2.4. Let $f : S^1 \to \mathbb{R}^2$ be a continuous map such that there are two points $a, b \in S^1$ such that $f(a) = f(b)$ and f is injective on $S^1 \setminus \{a\}$. Prove that $\mathbb{R}^2 \setminus f(S^1)$ has exactly three connected components.

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Problem set 5; October 31, 2000

Due on Tuesday, November 12

33. Calculate the fundamental group of the standard model for the orientable surface S_g of genus $g \geq 1$: the regular $4g$-gon with pairs of sides identified like this (a cyclic order is assumed): for $k = 0, 1, 2 \ldots, g-1$ the side $4k+1$ is identified with $4k+3$ and $4k+2$ with $4k+4$; in both cases the identification changes direction.

34. Prove that the fundamental groups from the previous problem for different values of g are pairwise nonisomorphic.

35. Describe rigorously a differentiable structure on the surface from Problem 33 which coincides with the standard structure outside of the vertices.

36. Consider the regular $2m$-gon with the pairs of opposite sides identified by parallel translations. Prove that the resulting space is homeomorphic to $S_{[m/2]}$.

37. Prove that the fundamental group of any finite one–dimensional connected simplicial complex is free.

38. Find the number of generators for the fundamental groups of one–dimensional simplcial complexes formed by the vertices and edges of the
(i) tetrahedron;
(ii) octahedron.

39. Find the fundamental group of the "necklace": the disjoint union of $m \geq 2$ spheres $S^{(1)}, \ldots, S^{(m)}$ with pairs of point identified in a cyclic order: $q_1 \in S^{(1)}$ is idenified with $p_2 \in S^{(2)}$, $q_2 \in S^{(2)}$ with $p_3 \in S^{(3)}$ etc, until $q_m \in S^{(m)}$ is identified with $p_1 \in S^{(1)}$.

40. Consider the normal subgroup in the free group with two generators F_2 generated by the commutators of the generators. Describe rigorously the corresponding covering of the figure eight.

41. Show that the "flat" torus $S^1 \times S^1$ is equivalent as a differentiable manifold to the "bagel" torus embedded into $\mathbb{R}^3$.

42. Consider a compact surface S in $\mathbb{R}^3$ given by a single equation $F(x_1, x_2, x_3) = 0$ where F is a differentiable function for which 0 is not a critical value. *Assume* that for any two sufficiently close points in S there is a unique shortest smooth curve connecting these points. Prove that S is a simplicial polyhedron, *i.e.* that is allows a simplicial decomposition (representation as a simplicial complex).

ADDITIONAL OPTIONAL PROBLEMS; submit solutions by November 28

A15. Find the number of generators for the fundamental groups of one–dimensional simplcial complexes formed by the vertices and edges of the icosahedron.

A16. Prove the assertion of Problem 42 without assuming existence of the unique shortest curve.

A17. Find a "politically correct" projection of the sphere with two poles removed to the cylinder $C = S^1 \times (0,1)$ mentioned in class, *i.e.* a differentiable map which sends parallels into "horizontal" circles, meridians into the "vertical" intervals, and preserves the area.

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THIS IS THE LAST PROBELM SET FOR FALL SEMESTER

Problem set 6 ; November 21

Due on Tuesday December 5

43. Prove that adding two more Mobius caps to the sphere with several handles and $m \geq 1$ Mobius caps is equivalent to adding a handle.

44. A surface M (two–dimensional differentiable manifold) is *orientable* if one can choose the direction of rotation in each tangent space T_xM which changes continuously from point to point

Prove that a nonorientable surface has an orientable double cover.

Hint: Look how orientation changes along a path.

45. Prove that the sphere with $m \geq 1$ Mobius caps is nonorientable.

46. Find the orientable double cover for the sphere with m Mobius caps.

47. Suppose that a simplicial polyhedron represents a compact surface. Prove that its second Betti number is equal to one if the surface is orientable and zero otherwise.

48. Prove that two–dimensional differentiable manifold is orientable if and only if it has an atlas such that all coordinate changes in the intersections of charts have positive Jacobian.

49. Prove that compact two–dimensional differentiable manifold is orientable if and only if in has a positive *volume element*: an antisymmetric diffrerential two form which does not vanish at any point.

50. Prove that the only surfaces which appear as factors of Euclidean plane with respect to free totally discontinuous actions of group of isometries are the cylinder, the Mobius strip, the torus and the Klein bottle.

Hint: Find the subgroup of translations and show that it has no more than two generators.

51. Consider the regular octagon with the pairs of opposite sides idnetified by translations. Desrcibe the structure of *one dimesnsiomal complex manifold* on it which coincides with the standard structure inside.

52. Consider the *ideal triangle* T on the hyperbolic plane with all its vetrices p, q and r on the real line. Consider the group of isometries generated by parabolic

transformations P_p which take the line pq into pr and P_q which takes the lines qp into qr.

Prove that this is a free group with two generators actig freeley and properly discontinuosly on the hyperbolic plane.

Prove that the factor space is homeomorphic to the sphere with three points removed.

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FINAL EXAMINATION

Saturday, December 9 ,2000

Do two problems from each section.

SECTION 1: **Fundamental group and homology.**

1.1. Consider the bouquet of three circle and the subgroup of its fundamental group generated by two of its three generators. Describe rigorously the corresponding covering space and covering map.

1.2. Consider the subgroup of the free group with two generators a and b generated by a^2 and b^3. describe the corresponding covering space of the figure eight.

1.3. Consider the following schematic representation of a small house:

$$H = \{(x, y, z) \in R^3 : 0 \leq x \leq 3,\ 0 \leq y \leq 2,\ 0 \leq z \leq 2$$

$$\text{and } x \in \{0, 1, 2, 3\},\ \text{or } y \in \{0, 1, 2\},\ \text{or } z \in \{0, 1, 2\}\}.$$

The set R is obtained from H by identifying coordinates mod 3.

Calculate the homology groups of R. (You may use any convenient triangulation.)

1.4. Give an example of two connected finite simplicial complexes which have isomorphic homology groups in all dimensions but nonisomorphic fundamental groups.

SECTION 2: **Surfaces and elements of differentiable manifolds.**

2.1. Describe a smooth embedding of the Klein bottle into $\mathbb{R}^4$.

2.2. Construct a Morse function with one maximum, one minimum and one saddle on the projective plane.

2.3. Attaching *an inverted handle* to an orientable surface is the following procedure: cut two small holes in the surface and identify the boundaries with two boundary components of the cylinder using opposite orientations on the cylinder for the two circles with the same orientation on the surface.

Identify the sphere with two inverted handles attached with one of the standard models: the sphere with several handles, or the sphere with several handles and one or two Mobius caps.

2.4. Suppose that a compact connected m-dimensional differentiable manifold M has an antisymmetric differentiable m form which vanishes on a contractible set. Prove that M is orientable.

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Problem set 7; January 23, 2001

Due on Monday February 5

53. Cosider the Mobius strip as a one-dimensional vector bundle over the circle. Prove that the Whitney sum of this bundle with itself is a trivial bundle (i.e. equivalent to the direct product $S^1 \times \mathbb{R}^2$.

54. An n-dimensional differentiable manifold is orientable if and only if its tangent bundle has a $SL(n, \mathbb{R})$ reduction.

55. Suppose that a two–dimensional vector bundle is orientable and has one non-vanishing section. Show that it is trivial.

56. Consider a vector bundle whose structure group is the group of upper–triangular matrices with positive diagonal elements. Show that is has a trivial reduction.

57. Prove that the tangent bundle to S^3 is trivial.
    Hint: Consider linear vector fields on S^3.

58. Show that the sphere S^{2n+1} is the total space of a principal S^1 bundle over the complex projective space $\mathbb{P}(n)$.

59. Show that the Hopf fibration of S^3 over S^2 with the fiber S^1 (which is a particular case of the construction of Problem 58 for $n = 1$) is not trivial, i.e. it is not the direct product.
    Hint: Use fundamental groups.

60. Prove that the fundamental group of the total space of a vector bundle is isomorphic to a subgroup of the fundamental group of the base.

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Problem set 8; February 9, 2001

Due on FRIDAY February 16

61. Consider the one-parameter groups of rotations of the standard unit sphere S^2 in $\mathbb{R}^3$ around the three coordinate axes. Let $v_x,\ v_y$ and v_z be their generating vector fields. Express the brackets of these vector fields as their linear combinations (such an expression is not unique but you may try to look for an elegant one, e.g for the coefficient to be constant).

62. Construct three linear linearly independent non-vanishing vector fields on the standard unit sphere in $\mathbb{R}^4$ and calculate their Lie brackets.

Hint: You may try to find an analogy with the setting of the previous problem, or, if you heard about quaternions, think about S^3 as the set of quaternions of the unit norm.

63. M be a two–dimensional differentiable manifold, u and v be two smooth vector fields on M which are linearly independent at a point $p \in M$. Prove that there exists a coordinate system (x_1, x_2) in a neighborhood of p and positive smooth functions α and ρ so that locally near p,

$$u = \alpha \frac{\partial}{\partial x_1},\ v = \rho \frac{\partial}{\partial x_2}.$$

64. Given a simple closed parametrized curve γ on a smooth manifold M, i.e. an embedding $S^1 \to M$, there exists a vector field on M for which γ is an orbit.

Hint: Use implicit function theorem and partition of unity.

65. Consider the group H of 3×3 upper-diagonal matrices with the units on the diagonal (the Heisenberg group). This group has natural coordinates $(x_{12}, x_{13}, x_{,23})$ and it acts on itself by left translations. Let $v_{12}, v_{13}, v_{,23}$ be the left-invariant vector-fields on H with the values at the identity $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ correspondingly. Calculate these vector fields and their brackets in coordinates $(x_{12}, x_{13}, x_{,23})$.

66. Calculate the dimension of the space $\mathcal{S}^m(\mathbb{R}^n)$ of symmetric m-forms on $\mathbb{R}^n$.

67. The *kernel* $\mathrm{Ker}(\omega)$ of a skew-symmetric bilinear form $\omega \in \bigwedge^2(L)$ is defined by

$$\mathrm{Ker}(\omega) = \{v \in L : \omega(v,u) = 0 \forall u \in L\}.$$

Call $r(\omega) = \dim L - \mathrm{d}im\, Ker(\omega)$ the *rank* of ω. Prove that $r(\omega)$ is always an even number and that for a form of rank $2k$ in n–dimensional space there exists a coordinate system such that for $(u = (u_1, \ldots, u_n),\ v = (v_1, \ldots, v_n)$,

$$\omega(u, v) = \sum_{i=1}^{k} u_{2i-1}v_{2i} - v_{2i-1}u_{2i}.$$

Hint: Reduce the problem to the maximal rank case and use induction in dimension.

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Problem set 9; February 26, 2001

Due on March 11

68. Let ω be a differential 1-form. Calculate the value of the exterior two form $d(\omega)$ on a pair of vector fields v_1, v_2. The result may include values on the vector fields and their brackets and derivatives of those values.

69. Let ω be a differential 2-form. Caltulate the value of an exterior 3-form $d(\omega)$ on a triple of vector fields. The result may include values on the vector fields and their brackets and derivatives of those values.

70. Prove that the forms $dx_1, \dots, dx_n$ form a basis in the first De Rham cohomology on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.
Hint. Use Poincaré lemma in $\mathbb{R}^n$ and integration over one-cycles.

71. Bredon p. 82 N5.

72. Bredon p. 82 N6.

73. Find a counterpart of the previous problem for the sphere embeddings: what is the minimal value of m such that given a smooth embedding $\phi : S^2 \to \mathbb{R}^m$ there exists a hyperplane such that the composition of ϕ with the orthogonal projection is still an embedding?

74. Prove the following stronger version of the Poincaré lemma. If ω is a closed form in $\mathbb{R}^n$ which vanishes in a neigborhood of the origin than $\omega = d\alpha$ where α vanishes in a (possibly smaller) neighborhood of x.

75. Prove without referring to de Rham Theorem that every closed k-form on the sphere S^n, where $0 < k < n$ is exact.
Hint. Use the previous problem.

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FIRST MID-TERM EXAMINATION

Friday, March 2 2001

Do two problems from each section.

SECTION 1

1.1. Prove that the tangent bundle TM of any compact manifold is *stably trivial* i.e. there exists another vector bundle F over M whose Whitney sum with TM is a trivial bundle.

1.2. Prove that any $\mathbb{R}^2$ bundle over the circle is either trivial or isomorphic to the Whitney sum of the Mobius strip bundle and the trivial bundle.

1.3. Prove that the fundamental group of a circle bundle over a sphere of dimension ≥ 2 is cyclic.

1.4. Prove that the tangent bundle to $S^n \times \mathbb{R}$ is trivial.

SECTION 2

2.1. Construct three linearly independent non-vanishing vector fields on the sphere S^{4n-1}.

2.2. Let M be a three–dimensional differentiable manifold, u and v be two smooth vector fields on M which are linearly independent at a point $p \in M$. Is it always true that there exists a coordinate system (x_1, x_2, x_3) in a neighborhood of p and positive smooth functions α and ρ so that locally near p,

$$u = \alpha \frac{\partial}{\partial x_1}, \ v = \rho \frac{\partial}{\partial x_2} \ ?$$

2.3. A differentiable two-form ω on a $2k$-dimensional manifold is called *symplectic* if any point has a neighborhood with coordinates $(x_1, \dots, x_k, y_1, \dots, y_k)$ such that locally $\omega = \sum_{i=1}^{k} dx_i \wedge dy_i$. Prove that a manifold which carries a symplectic form is orientable.

2.4. Prove that the wedge product induces a ring structure in the direct sum of the de Rham cohomology spaces, i.e. that if both ω_1 and ω_2 are closed then $\omega_1 \wedge \omega_2$ is closed, and if ω_1 is closed and ω_2 is exact, then $\omega_1 \wedge \omega_2$ is exact.

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Problem set 10; March 16, 2001

Due on Wednesday March 28

76. Give a detailed proof of the calculation of de Rham cohomolory ring for the torus $\mathbb{T}^n$.

77. Calculate the de Rham cohomolory ring for the real projective space $\mathbb{R}P(n)$.

78. Give an explicit construction of a basis in the first de Pham cogomology group (i.e. describe particular closed forms representing the cohomology classes) for the sphere with two handles using either the representation as a smooth "pretzel" in $\mathbb{R}^3$ or one of the polygonal representations.

79. Let M be a compact m-dimensional manifold and B be a subset of M diffeomorphic to the closed m-dimensional ball. Show that every closed differential k-form on M is cohomologous to a form which vanishes on B.

80. Define the *intersection index* in the first de Rham cogomology group of a compact orientable surface M by fixing an orientation on M and putting

$$\operatorname{int}(\omega_1, \omega_2) = \int_M \omega_1 \wedge \omega_2$$

Prove that the intersection index is correcly defined and that it is a skew-symmetric bilinear form on $H^1_{dR}(M)$.

81. Take M as the sphere with g handles, fix a basis in $H^1_{dR}(M)$ and calculate the intersection index form in that basis.

82. Define an intersection index in $H^2_{dR}(M)$ for a compact four-dimensional orientable manifold in a fashion similar to Problem 80 and calculate it for the natural basis for $M = \mathbb{T}^4$.

83. Prove that de Rham cohomology of the complex projective space $\mathbb{C}P(n)$ is nontrivial in even dimensions (i.e. dimension $2k,\ k = 0, \ldots, n$).

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Problem set 11 ; April 2, 2001

84 . (**Due on Wednesday April 4**) Calculate the cellular homology of the unit tangent bundle $T_1(S^2)$ using the cellular decomposition described in class of March 30.

Hint: The identification map between the boundaries of two solid tori is not cellular in the natural cell decompostion of the torus. Rather than refining the decomposition to make it cellular (which is possible but messy) see that the identification map is homotopic to a cellular map which is cufficient for the homology calculation

THE REST OF THE SET IS DUE IN WEDNESDAY APRIL 11

85. Consider the standard cellular decomposition of the three-dimensional torus $\mathbb{T}^3$. Let f be the linear map given by the matrix

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Describe explicitly a cellular approximation of the map f.

86. Prove that the three–dimensional sphere S^3 is homeomorphic the the factor–space of the union of two solid tori with the boundaries identified via the map $f(x, y) = (y, x) \pmod 1$. Explain the difference between S^3 and $T_1(S^2)$ using the difference in identification on the boundaries.

87. Show that not every 4×4 matrix appears as the induced map on the first de Rham cohomology group for a map of the sphere with two handles into itself

Hint: Use the intersection index from Problem 80.

88. Consider the self–map f of the two-dimensional projective space $\mathbb{C}P(2)$ give in the homogeneous coordinates by

$$f(z_1, z_2, z_3) = (z_1 z_2^2 + z_1^3,\ z_2^2 z_3,\ z_1 z_2 z_3).$$

Prove that it is a cellutar map with respect to the standard cellular decompoition (the 0-cell $z_2 = z_3 = 0$, the 2-cell $z_3 = 0$ and the four–cell) and calculate induced map on the homology groups.

Hint: Use the definitoion of the index based on the Sard Theorem.

89. Consider the identification space of of the union of two solid tori with the boundaries identified using the map $f(x, y) = (-y, x) \pmod 1$. Prove that this

space has a natural structure of a three–dimensional manifold, describe a cell decomposition and calculate the cellular homology.

90. Show that any connected CW complex allows a cell decomposition with one vertex.

91. Show that every connected simplicial complex which is a pseudomanifold allows a cell decomposition with a single cell of maximal dimension.

Hint: Start with a simplicial decomposition and modify it only in the maximal dimension by successively ”squeezing ” the ball of “toothpaste” into the simplices so that eventually it fills them all.

MATH 528: TOPOLOGY/GEOMETRY

A.Katok

SECOND MID-TERM EXAMINATION

Friday, April 13 2001

Do two problems from each section.

SECTION 1

1.1. Consider an orientable compact surface M^2 with an area form (a nondegenerate differentiable 2-form) Ω. Prove that a differential 1-form α is closed if and only if

$$\alpha = v \lrcorner \Omega,$$

where v is an area–preserving vector field.

1.2. Consider the linear map F_A of the torus $\mathbb{T}^n$ given by the integer matrix A with n different real eigenvalues $\lambda_1, \ldots, \lambda_n$. Find the eigenvalues of the map induced by F_A in the group $H^k_{dR}(\mathbb{T}^n)$.

1.3. Prove that if the intersection index of two non-zero elements $\alpha, \beta \in H^1_{dR}(\mathbb{T}^2)$ is equal to 0, then α and β are proportional.

1.4. Prove that on any compact differentiable manifold, orientable or not, there exists a *positive density form*, i.e a linear functional I on the space of continuous functions, continuous in the uniform topology, and such that for a function f which is equal to zero outside of a a coordinate neighborhood U,

$$I(f) = \int f(x_1, \ldots, x_n)\rho(x_1, \ldots, x_n)dx_1dx_2 \ldots dx_n,$$

where the weight function ρ is positive and differentiable.

SECTION 2

2.1. Consider the torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ with the standard cell decomposition (one 0-cell, three 1-cells, three 2-cells, and one 3-cell) which appears from the identification of the opposite sides of the unit cube. Find a necessary and sufficient condition for a linear map F_A of $\mathbb{T}^3$, i.e a map induced by a 3 matrix A with integer elements, to be homotopic to a cellular map which is a homeomorphism.

2.2. Prove that the fundamental group of a connected CW complex coincides with the fundamental group of its two-skeleton (i.e. the union of zero-, one- and two-cells.

2.3. Give an example of a compact metrizable space which allows a cellular decomposition, but not a triangulation.

2.4. Calculate the cellular homology of $S^2 \times \mathbb{R}P(2)$.

Ph. D. QUALIFYING EXAMINATION IN GEOMETRY AND TOPOLOGY

Saturday, MAY 5,2001

Do two problems from each section.

SECTION 1

1.1. Prove that the real line can be represented as uncountable union of disjoint subsets each of which is homeomorphic to the Cantor set.

1.2. Suppose X is a path–connected space whose fundamental group is S_3, the group of permutations of three symbols. How many nonhomeomorphic covering spaces does X have?

1.3. Find the fundamental group of the surface of the cube with interiors of all edges removed, i.e. the space which consists of the vertices and interiors of the faces of the cube.

1.4. Topological spaces X and Y are homotopically equivalent and X is Hausdorff. Is Y Hausdorff?

SECTION 2

2.1. Let $\gamma : S^1 \to \mathbb{R}P(2)$ be an injective null-homotopic continuous map. Prove that $\mathbb{R}P(2) \setminus \gamma(S^1)$ consists of two connected components one of which is homeomorphic to the disc and other is not.

2.2. The third Betti number of a simplicial complex is equal to three. What is the minimal number of vertices in the complex?

2.3. Construct a finite connected CW complex with the first homology group $\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$, the second homology group $\mathbb{Z}/6\mathbb{Z}$, the third homology group $\mathbb{Z}^2$, and the fourth homology group equal to zero.

2.4. Let X be a three–dimensional vector bundle over a compact base whose structure group is the group $O(2,1)$ of matrices preserving the quadratic form $x_1^2+x_2^2-x_3^2$. Prove that this bundle has a nontrivial one-dimensional subbundle.

SECTION 3

3.1. Does there exist a differentiable map $f : \mathbb{T}^2 \to S^2$ such that for an open set A of points in S^2 the preimage $f^{-1}(x)$ for each $x \in A$ contains exactly three points?

3.2. Consider the vector field

$$v = \sin(2\pi x_2)\frac{\partial}{\partial x_1}$$

on the standard two-dimensional torus. Prove that any vector field u which commutes with v, i.e. $[u, v] = 0$, is collinear with v.

3.3. Suppose M is a compact symplectic $2m$-dimensional manifold, i.e there exists a closed 2-form ω on M such that ω^m does not vanish. Prove that the de Rham cohomology of M in every even dimension from 2 to $2m$ is not zero.

3.4. Prove that the group $SO(4)$ of orthogonal 4×4 matrices with determinant one is homeomorphic to $S^3 \times \mathbb{R}P(3)$.

Ph. D. QUALIFYING EXAMINATION IN GEOMETRY AND TOPOLOGY

Monday, August 13,2001

Do two problems from each section.

SECTION 1

1.1. Prove that for any two points x and y in the Cantor set C there exists a homeomorphism $f : C \to C$ such that $f(x) = y$.

1.2. Consider the following subsets of the plane:

$$A = \{x \geq 0,\ y = |\sin x|\} \text{ and } B = \{0, 0\} \cup \{x > 0,\ y = |x \sin \frac{1}{x}|\}.$$

Each set is path-connected and contains countably many loops joint to each other. Are the fundamental groups of A and B isomorphic?

1.3. Let G be the subgroup of F_3, the free group with three generators which consists of all elements for which the shortest representation as a word composed of generators and their inverses has even length. Describe the corresponding covering space.

1.4. Prove that the factor space of the complex projective plane $\mathbb{C}P(2)$ by the complex conjugation is homeomorphic to the sphere S^4.

SECTION 2

2.1. Prove that any simplicial decomposition of the two-dimensional torus contains at least ten two-dimensional simplices.

2.2. Denote the vectors of the standard basis in $\mathbb{R}^4$ by e_1, e_2, e_3, e_4 and let $e_5 = (1/4, 1/4, 1/4, 1/4)$. Let

$$X = \{x \in \mathbb{R}^4 :\ x = \alpha_1 e_{i_1} + \alpha_2 e_{i_2} + \alpha_3 e_{i_3},\ \alpha_1 + \alpha_2 + \alpha_3 = 1, \alpha_i \geq 0,\ i = 1, 2, 3\}.$$

where $\{i_1, i_2, i_3\}$ is any subset of $\{1, 2, 3, 4, 5\}$. In other words, X is the soap bubble on the wire consisting of the edges of a regular tetrahedron and the spikes connecting the center with the vertices.

The set X comes with a natural simplicial decomposition. Calculate the homology groups of X.

2.3. Construct a finite CW complex such that its homology groups in dimension $1, \ldots m$ are given finite abelian groups $G_1, \ldots, G_m$ and higher homology groups are trivial.

2.4. Consider a locally trivial bundle B with compact base whose fibers are $\mathbb{R}^n$ and the structure group is the group of affine transformations. Prove that there is a reduction of the structure group to the group of Euclidean isometries of $\mathbb{R}^n$.

SECTION 3

3.1. Does there exist a differentiable map $f : \mathbb{T}^4 \to \mathbb{T}^4$ of the four-dimensional torus $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$ into itself such that $f^*[dx_1 \wedge dx_2] = [dx_2 \wedge dx_3]$ and $f^*[dx_1 \wedge dx_3] = [dx_1 \wedge dx_4]$? Here $[\cdot]$ denotes the de Rham cogomology class and f^* is the map induced by f on the de Rham cohomology groups.

3.2. Show that there exists a Morse function on the real projective space $\mathbb{R}P(n)$ with exactly one critical point of Morse index k for $k = 0, 1 \ldots, n$.

3.3. Prove that the tangent bundle to the projective space $\mathbb{R}P(3)$ is trivial.

3.4. For a vectorfield v on the standard flat torus $\mathbb{T}^2$ let Jv be the vectorfield obtained by the rotation by $\pi/2$ in the positive direction. Assume that v is smooth and that both vectorfields v and Jv preserve the area form $dx_1 \wedge dx_2$. Prove that v has constant coefficients:

$$v = \alpha \frac{\partial}{\partial x_1} + \beta \frac{\partial}{\partial x_2}.$$