

MATH 527: TOPOLOGY/GEOMETRY

A.Katok

PROBLEM SET # 1

TOPOLOGICAL SPACES

Due on Monday 9-12-94

1. Find all different topologies (up to a homeomorphism) on the sets consisting of 2 and 3 elements.
2. Prove that the set of squares of rational numbers is dense in the set of all non-negative real numbers.
3. Prove that for any set A in a topological space $\partial\bar{A} \subset \partial A$ and $\partial(\text{Int } A) \subset \partial A$. Give an example when all these three sets are different.
4. We say that a topological space (X, \mathcal{T}) satisfies (T_1) separation axiom (or simply is a (T_1) -space) if for any two different points x and y there exists an open set U which contains x and does not contain y . Prove that (X, \mathcal{T}) is a (T_1) -space if and only if any set consisting of one point is closed.
5. Find among examples given in class a topological space which is a (T_1) -space, but not a (T_2) (Hausdorff) space.
6. Prove that the product of countably many separable topological spaces with the product topology is separable.
7. Prove that a topological space (X, \mathcal{T}) is connected if and only if any continuous function from X to the set of integers (with discrete topology) is constant.
8. Prove that \mathbb{R} (the real line) and \mathbb{R}^2 (the plane with the standard topology) are not homeomorphic. *Hint:* Use the notion of a connected set.

ADDITIONAL PROBLEMS

- A1. A topological space (X, \mathcal{T}) is called *regular* (or (T_3) -space) if for any closed set $F \subset X$ and any point $x \in X \setminus F$ there exist disjoint open sets U and V such that $F \subset U$ and $x \in V$. Give an example of a Hausdorff topological space which is not regular.
- A2. Prove that a topological space (X, \mathcal{T}) is connected if and only if any continuous function $f : X \rightarrow K$, where K is the Cantor set, is constant.
- A3. A point x in a topological space is called *isolated* if the one-point set $\{x\}$ is open. Prove that any compact separable Hausdorff space without isolated points contains a closed subset homeomorphic to the Cantor set.

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PROBLEM SET # 2

EXAMPLES OF TOPOLOGICAL AND METRIC SPACES

NN9–12 are due on Monday 9-19-94; NN13–16 and A4–A7 are due on Monday 9-26-94

9. Find all different topologies (up to a homeomorphism) on a set consisting of 4 elements which make it a connected topological space.

10. Let X, Y be two topological spaces $f : X \rightarrow Y$ be a continuous map,

$$\text{graph } f = \{(x, f(x)) \in X \times Y; x \in X\}.$$

Prove that graph f with the topology induced from $X \times Y$ is homeomorphic to X .

11. Prove that the set $[0, 1] \times [0, 1] \setminus K \times K$, where K is the standard Cantor set, is path-connected.

12. Let $\{t_n\}$, $n = 1, 2, \dots$ be a sequence of positive numbers such that the series $\sum_{n=1}^{\infty} t_n$ converges. Let for $\omega, \omega' \in \Omega_2$

$$d(\omega, \omega') = \sum_{n=1}^{\infty} t_n |\omega_n - \omega'_n|.$$

Prove that this formula defines a metric on the space Ω_2 which generates the product topology.

13. Consider the product Ω_m , $m \geq 2$ of countably many m -point sets with discrete topology. Prove that Ω_m provided with the product topology is a Cantor space i.e. is homeomorphic to the standard Cantor set K .

14. Let the group \mathbb{R} act on \mathbb{R}^2 by

$$t(x_1, x_2) = (x_1, x_2 + x_1 t).$$

Prove that the factor-space with the factor topology is not Hausdorff but it is a union of two disjoint subsets each of which is a Hausdorff topological space. Prove also that the factor space is a T_1 -space (see problem 4).

15. For a given prime number p define the p -adic norm $\| \cdot \|_p$ on the field \mathbb{Q} of rational numbers by

$$\|r\|_p = p^{-m}, \text{ if } r = p^m \frac{k}{l}, \text{ where } mk, l \in \mathbb{Z} \text{ and } k \text{ and } l \text{ are relatively prime with } p.$$

Prove that the distance function $d(r, r') = \|r - r'\|_p$ defines a metric on \mathbb{Q} . The completion of \mathbb{Q} in that metric is called *p-adic numbers* and the completion of \mathbb{Z} is called *p-adic integers*. Prove that the space of *p-adic integers* is a Cantor space and the space of *p-adic numbers* is homeomorphic to the disjoint union of countably many Cantor spaces. *Hints:* Use the fact that integers lie in the unit ball around zero. Use Problem 13.

16. Define the *profinite topology* on the group \mathbb{Z} of integers as the weakest topology in which any arithmetic progression is an open set. Let \mathbb{T}^∞ be the product of countably many copies of the circle with the product topology. Define the map $\varphi : \mathbb{Z} \rightarrow \mathbb{T}^\infty$ by

$$\varphi(n) = (\exp(2\pi in/2), \exp(2\pi in/3), \exp(2\pi in/4), \exp(2\pi in/5), \dots)$$

Show that the map φ is injective and that the topology induced on $\varphi(\mathbb{Z})$ coincides with profinite topology.

ADDITIONAL PROBLEMS

A4. Show that the closure of $\varphi(\mathbb{Z})$ as in problem 16 is homeomorphic to the Cantor set. Introduce a translation-invariant metric on \mathbb{Z} which generates the profinite topology and such that Cauchy sequences in that metric are exactly the sequences whose images under φ converge in \mathbb{T}^∞ .

A5. Consider the weakest topology in the set \mathbb{R} or real numbers such that for any $t \in \mathbb{R}$ the function $x \rightarrow \exp(itx)$ is continuous. Prove that this topology is not metrizable.

A6. Prove that any compact metrizable topological space is homeomorphic to a closed (and hence compact) subset of the *Hilbert cube*, i.e. the product of countably many unit intervals with the product topology (universality of the Hilbert cube).

A7. A metric space X is called *locally path connected* if for any $\epsilon > 0$ there exists $\delta > 0$ such that any two points at a distance less than δ can be connected by a path contained in a ball of radius ϵ

Prove that for any compact path connected, and locally path connected subset X of the plane \mathbb{R}^2 there exists a continuous map $f : [0, 1] \rightarrow \mathbb{R}^2$ whose image coincides with X (Generalized Peano curve).

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PROBLEM SET # 3

COMPACT AND COMPLETE SPACES; TOPOLOGICAL GROUPS

due on Wednesday 10-12-94

17. Prove that any separable metric space has a countable base.
18. Prove that for a metric space compactness is equivalent to sequential compactness: Every sequence contains a converging subsequence. *Hint:* Use previous problem.
19. Let X be a compact Hausdorff space with a countable base (and hence metrizable). Prove that the topology in the space $\mathcal{F}(X)$ of all closed subsets of X induced by the Hausdorff metric does not depend on the metric in X defining the given topology.
20. A metric space X is called *precompact* if for any $\epsilon > 0$ it can be covered by finitely many ϵ -balls.
 Prove that the completion of a metric space X is compact if and only if X is pre-compact.
21. Let the *weak topology* in the Hilbert space $l^2(\mathbb{R})$ be the weakest topology in which all maps $f : l^2(\mathbb{R}) \rightarrow \mathbb{R}$ of the form
- $$f(x) = \sum_{n=1}^{\infty} a_n x_n, \text{ for some } (a_1, a_2, \dots) \in l^2(\mathbb{R}).$$
- are continuous. Prove that the weak topology is weaker than the standard (norm) topology i.e that there are open sets in the norm topology which are not open in the weak topology.
22. A *topological group* is a group G endowed with a topology such that the group multiplication and taking inverse are continuous operations, i.e. the maps
- $$G \times G \rightarrow G : (g_1, g_2) \rightarrow g_1 g_2 \text{ and } G \rightarrow G : g \rightarrow g^{-1}$$
- are continuous. Two topological groups are isomorphic if there exists a group isomorphism between them which is also a homeomorphism.
 Prove that the space Ω_2 with the coordinate-wise modulo 2 addition as the group operation and the product topology is a topological group.
23. Consider the group $SL(2, \mathbb{R})$ of all 2×2 matrices with determinant one with the topology induced from the coordinate embedding into \mathbb{R}^4 . Prove that it is a topological group.
24. Prove that the addition and multiplication can be extended in a unique way from the rationals and non-zero rationals correspondingly to the set of p -adic numbers (problem 15) and non-zero p -adic numbers correspondingly so that the topology of problem 15 makes those into topological groups.

ADDITIONAL PROBLEMS

A8. Prove that any closed convex bounded set in $l^2(\mathbb{R})$ (e.g. any closed ball) is compact in weak topology.

A9. Prove that any set in $l^2(\mathbb{R})$ compact in the weak topology is closed and bounded. Give an example of a closed bounded set in $l^2(\mathbb{R})$ which is not compact in the weak topology.

A10. Give an example of a compact metrizable path-connected topological space X such that no point of X has a connected neighborhood.

A11. Consider the metric on \mathbb{Z} defining the profinite topology as in Problem A4. Show that addition can be extended in a unique way to the completion to make it into a topological group which is isomorphic as topological group to the product of the groups of p -adic integers for $p = 2, 3, 5, \dots$

A12. Consider the following subgroup S of \mathbb{T}^∞ , the product of countably many copies of the circle: $S = \{(z_1, z_2, z_3, \dots) : z_n^2 = z_{n-1}, n = 2, 3, \dots\}$ with the topology induced from \mathbb{T}^∞ . Prove that as topological space S is connected but not path-connected.

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PROBLEM SET # 4

MISCELLANEOUS GENERAL TOPOLOGY

due on Wednesday 10-26-94

25. Prove that on the real line \mathbb{R} there are uncountably many different (non-equivalent) complete uniform structures compatible with the standard topology

Hint: Use different metrics.

26. Prove that for any natural number n the standard n -dimensional simplex

$$\sigma^n = \{(x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{k=1}^{n+1} x_k = 1, x_k \geq 0, k = 1, \dots, n\}$$

is homeomorphic to the closed unit ball in \mathbb{R}^n

27. Consider the unit sphere in \mathbb{R}^n as a homogenous space of the group $SO(n)$ of orthogonal matrices with determinant one. Prove that the factor-topology coincides with the standard topology induced from \mathbb{R}^n .

28. Let X be a compact Hausdorff space. Prove that the space of continuous maps from X to the unit interval is compact if and only if X contains finitely many elements.

29. Prove that for any natural number k the space $C^k(\mathbb{R}^2)$ of all k times continuously differentiable functions of two real variables (with the topology of uniform convergence on compact sets of the functions and all partial derivatives of order up to k and the corresponding uniform structure) is complete.

30. Prove that the figure eight (i.e. the union of two circles with one common point) is not contractible.

31. Prove that the product of a finite or countable collection of contractible spaces is contractible.

32. A *path-connected component* of a topological space X is a maximal path-connected subset of X . Prove that any space can be decomposed in a unique way into path-connected components.

ADDITIONAL PROBLEMS

due on Wednesday 11-2-94

A13. Describe uncountably many different *incomplete* uniform structures on the real line \mathbb{R} compatible with the standard topology.

A14. Let $G = SL(2, \mathbb{R})$ be the topological group of all 2×2 matrices with determinant one. Consider the subgroup $H = SL(2, \mathbb{Z}) \subset SL(2, \mathbb{R})$ of all matrices with integer entries. Prove that the homogenous space G/H with the factor-topology is normal, locally compact but not compact.

A15. Construct a continuous map from the unit interval onto Hilbert cube (Infinite-dimensional Peano curve). Try not to do an explicit construction from the scratch but use existing examples instead.

A16. Prove that in the space $C([0, 1])$ of continuous functions on the unit interval the set of functions which are monotone on some interval has first category.

A17. A *one-dimensional complex* is a topological space which consists of a finite or countable union of sets (*edges*) each of which is homeomorphic to the unit interval with disjoint interiors and such that any endpoint of any edge belong only to finitely many edges. A *loop* is a collection of edges $\{E_1, \dots, E_n\}$ such that one end point of E_1 is also an endpoint of E_n , the other endpoint of E_1 is an endpoint of E_2 , the other endpoint of E_2 is an endpoint of E_3 etc. A complex is a *tree* if it does not contain any loops. Prove that one-dimensional complex is contractible if and only if it is a tree.

A18. Describe in detail the path-connected components of the topological space of Problem A12 (dyadic solenoid). In particular prove that every path-connected component is dense.

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MID-TERM EXAMINATION

GENERAL TOPOLOGY

Saturday 10-22-94

Do one problem from each of Sections 1 and 2 and two problems from Section 3.

SECTION 1

1.1. Let G be the group of 3×3 upper-triangular matrices whose diagonal entries are all equal to one. Consider the action of G on \mathbb{R}^3 by linear transformations. Prove that the factor space is T_1 . Find minimal number of Hausdorff subsets into which this factor-space can be divided.

1.2. Consider the action on \mathbb{R}^2 of the group generated by the rotation about the origin by the angle $\frac{2\pi p}{q}$ where p and q are positive integers. Prove that the factor-space is homeomorphic to \mathbb{R}^2 .

SECTION 2

2.1. Prove that there exists a continuous map of the unit interval $I = [0, 1]$ onto the unit ball D^3 in the three-dimensional Euclidean space.

2.2. Prove that the space of continuous maps from the real line to the unit interval is separable in the compact-open topology (uniform convergence on compact sets) and is not separable in the uniform topology.

SECTION 3

3.1. Prove that the set of all irrational numbers on the real line is not of type F_σ , ie that it can not be represented as a union of countably many closed sets.

3.2. Prove that the product of countably many finite spaces containing more than one point each and each provided with the discrete topology is a Cantor space.

3.3. Prove that the countable product of locally compact topological spaces $X_n, n = 1, 2, \dots$ is locally compact if and only if all but finitely many of the spaces X_n are compact.

3.4. Consider Zariski topology in \mathbb{R}^2 ie the complements of the zero sets of polynomials of two variables form a base of open sets. Prove that \mathbb{R}^2 with this topology is T_1 not Hausdorff and path-connected.

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PROBLEM SET # 5

HOMOTOPY EQUIVALENCE, FUNDAMENTAL GROUP

due on Friday 11-11-94

33. The (open) *Mobius strip* is the factor (orbit space) of \mathbb{R}^2 by the action of the group generated by integer translations along the y -axis and the transformation $T : (x, y) \rightarrow (-x, y + 1)$. Prove that the Mobius strip is homotopically equivalent to the circle.
34. Prove that any convex set in \mathbb{R}^n lies inside a certain affine subspace and contains an open ball in that subspace.
35. Prove that any convex set in \mathbb{R}^n is contractible.
36. Prove that the fundamental group of the Cartesian product of two path-connected topological spaces is isomorphic to the direct product of their fundamental groups.
37. Find the fundamental group of the figure eight.
38. Prove that any contractible space is path-connected.
40. Prove that the open cylinder with one point removed and the torus \mathbb{T}^2 with one point removed are homotopically equivalent and calculate the fundamental group of those spaces.
41. The *projective plane* is the factor-space of the two-dimensional sphere where pairs of opposite point are identified. Prove that the projective plane is not contractible and is not homotopically equivalent to a sphere or a torus of any dimension. *Hint:* Use fundamental groups.

ADDITIONAL PROBLEMS

due on Monday 11-28-94

A19. Prove the following special (in fact, a leading) case of the *Tychonov fixed-point theorem*: Every continuous map of the Hilbert cube into itself has a fixed point. You may use Brouwer fixed-point theorem.

A20. The *Klein bottle* is the factor (orbit space) of \mathbb{R}^2 by the action of the group generated by integer translations along the x -axis and the transformation $T : (x, y) \rightarrow (-x, y + 1)$. Prove that the Klein bottle is a topological manifold. Prove that it is not homotopically equivalent to the bouquet of $n \geq 1$ circles.

A21. Prove that the fundamental group of any compact topological manifold is finitely generated.

A22. Prove that the fundamental group of any one-dimensional complex (See Problem A18) is a free group with a finite or countable number of generators

A23. Prove that the unit sphere in the Hilbert space $l^2(\mathbb{R})$ is contractible.

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PROBLEM SET # 6

**DEFORMATION RETRACTS, FUNDAMENTAL GROUP,
COVERING SPACES.**

due on MONDAY 11-28-94

42. Give a detailed rigorous argument showing that the figure eight is a strong deformation retract of the "basic pretzel" (the solid double torus).
43. Consider the quadric in \mathbb{R}^n given by the equation $\sum_{i=1}^k x_i^2 - \sum_{i=k+1}^n x_i^2 = 1$. Prove that it has a $k - 1$ -dimensional sphere as a deformation retract.
44. Calculate the fundamental group of the topological group $SL(2, \mathbb{R})$.
45. Recall that the *complex projective space* $\mathbb{C}\mathbb{P}(n)$ is the space of all complex lines passing through the origin in the $n + 1$ -dimensional complex space \mathbb{C}^{n+1} , or, equivalently, the factor of \mathbb{C}^{n+1} minus the origin with respect to the action of the multiplicative group of non-zero complex numbers by the scalar multiplication. The natural embedding $\mathbb{C}^n \subset \mathbb{C}^{n+1}$ generates an embedding $\mathbb{C}\mathbb{P}(n - 1) \subset \mathbb{C}\mathbb{P}(n)$. Prove that $\mathbb{C}\mathbb{P}(n - 1)$ is a strong deformation retract of $\mathbb{C}\mathbb{P}(n)$ with one point deleted.
46. Prove that $\mathbb{C}\mathbb{P}(n)$ is simply connected. *Hint:* Use previous problem and induction in dimension.
47. Calculate the fundamental group of the real projective space $\mathbb{R}\mathbb{P}(n)$.
48. The *Klein bottle* is the factor (orbit space) of \mathbb{R}^2 by the action of the group generated by integer translations along the x -axis and the transformation $T : (x, y) \rightarrow (-x, y + 1)$. Prove that the Klein bottle is a topological manifold. Prove that it is not homotopically equivalent to any of the following spaces: point, sphere of any dimension, torus of any dimension.
49. Prove that any covering space of the Klein bottle is homeomorphic to one of the following spaces: \mathbb{R}^2 , \mathbb{T}^2 , open cylinder, Mobius strip and Klein bottle.
50. Describe the covering space of the figure eight corresponding to the commutant of the fundamental group.

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PROBLEM SET # 7

SIMPLICIAL COMPLEXES AND SIMPLICIAL HOMOLOGY

due on Friday 12-9-94

51. Describe a simplicial decomposition of the sphere with n handles.
52. Describe a simplicial decomposition of the real projective space $\mathbb{R}P^n$.
53. Calculate the first and second homology groups of the Klein bottle. Hint: Use Poincaré-Hurewicz Theorem.
54. Prove that the homology groups of a bouquet of finitely many connected simplicial complexes are direct products of the corresponding homology groups.
55. Using problem 52 calculate all homology groups of the space $\mathbb{R}P^3$.
56. Find the minimal number of vertices in a simplicial complex S such that $H_1(S) = \mathbb{Z}^6$.

58.

Adding two Möbius caps to at least one is equivalent to adding a handle.

Equivalence of two octagon representation of the surface of genus two.

Covering between the surfaces (converse use Euler Theorem)

51. Let $m \leq n$. Consider the bouquet of n circles B_n with the common point p and let $F_m \subset F_n = \pi_1(B_n, p)$ be the subgroup generated by the first m generators. Describe the covering of B_n corresponding to this subgroup.

Topics:

1. Homotopy invariance of the fundamental group
2. Sphere is not a deformation retract of the ball implies Brouwer fixed point theorem (Thus already know in dimension two)
3. Sketch of proof. Enough to show S^n not contractible.. For that sufficient to show identity map not homotopic to zero. Homotopy may be approximated by a smooth homotopy. .But then the push-forward of the volume has locally constant integral. (See section 8.2) Alternatively, refer to Sard.
4. Covering spaces. Principle of covering homotopy. Examples. Deck transformations. Relations between fundamental groups. construction of a cover by a subgroup.
5. Universal cover. Uniqueness, Characterization of covers. Example: universal cover of figure eight
6. Covers and properly disconnected free actions. Examples.
7. Topology of surfaces. Handles and Mobius caps. Orientability. Calculation of the fundamental group. Euler Theorem.
8. Simplicial complexes and simplicial decompositions of topological spaces. Boundary operator. Chains, cycles, boundaries. Homology of a simplicial complex. Examples.

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FINAL EXAMINATION

Monday 12-12-94

Do one problem from each section.

SECTION 1

1.1. Consider the *weak topology* in the n -dimensional complex space \mathbb{C}^n , i.e. the weakest topology which make any linear function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ continuous. Prove that it coincides with the standard (Euclidean) topology.

1.2. Let X be a compact metric space with the distance function d . Introduce the following metric in the space $Hom(X)$ of all homeomorphisms of X onto itself.

$$dist(f, g) = \max_{x \in X} \max(d(x, fg^{-1}x), d(x, gf^{-1}x))$$

Prove that $Hom(X)$ is a complete topological group.

SECTION 2

2.1. Let $m \leq n$. Consider the bouquet of n circles B_n with the common point p and let $F_m \subset F_n = \pi_1(B_n, p)$ be the subgroup generated by the first m generators. Describe the covering of B_n corresponding to this subgroup.

2.2. Let N be the real line to which circles are attached at all integer points. Prove that there exists a covering map $p : N \rightarrow B_2$ which is a normal cover. Describe a subgroup of $F_2 = \pi_1(B_2)$ corresponding to that cover and the group of deck transformations.

SECTION 3

3.1. Let X_k be the unit disc whose boundary points are identified if their arguments differ by a multiple of $2\pi/k$. Calculate the fundamental group of X_k .

3.2. Calculate the fundamental group of the Klein Bottle with two points removed.

Note: You can pick any two different points you want. You do not have to prove that the result does not depend on the choice of points.

SECTION 4

4.1. Let S be a simplicial polyhedron, $p_1, p_2, \dots, p_n \in S$. Prove that there exists a simplicial decomposition of S which has the points p_1, p_2, \dots, p_n among its vertices.

4.2. Construct a compact set $A \subset \mathbb{R}^2$ which is a union of two simplicial polyhedra but is not a simplicial polyhedron itself.

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PROBLEM SET # 1

SIMPLICIAL COMPLEXES AND SIMPLICIAL HOMOLOGY

due on Thursday 2-9-95

1. Let K be an n -dimensional simplicial complex, K' its subcomplex consisting of all simplexes of dimension less than n . Calculate the relative homology groups $H_i(K, K')$.
2. Let K be a connected simplicial complex, L its zero-dimensional subcomplex, i.e. L consists of several, say m , vertices of K . Prove that for $i \geq 2$, $H_i(K, L) = H_i(K)$ and $H_1(K, L) = H_1(K) \oplus \mathbb{Z}^{m-1}$.
3. Prove that the direct product of two simplicial polyhedra is a simplicial polyhedron.
4. Prove that the first homology group of the direct product of two simplicial polyhedra is isomorphic to the direct product of their first homology groups.
5. Let P be a compact convex polyhedron in \mathbb{R}^3 ; V , E and F be the numbers of its vertices, edges and faces correspondingly. Prove the *Euler Theorem*: $V - E + F = 2$
Hint: You may use the fact that homology groups of a simplicial polyhedron are independent of a simplicial decomposition.
6. Let K be an n -dimensional simplicial complex with the following property: the union of interiors of its n - and $(n - 1)$ - dimensional simplexes is connected and every $(n - 1)$ - dimensional simplex belongs to at most two n - dimensional simplexes. Prove that $H_n(K)$ is equal to either 0 or \mathbb{Z} .
7. Under the conditions of the previous problem show that $H_n(K) = 0$ if one of the following conditions holds (i) the total number of n - and $(n - 1)$ - simplexes in K is infinite; or (ii) some $(n - 1)$ - simplex belongs to the boundary of exactly one of n simplexes. Give an example when none of these conditions hold but still $H_n(K) = 0$.
8. Calculate the Euler characteristic of the sphere with m handles by counting the numbers of simplexes in a simplicial decomposition (cf. Problem 51). Show that for the second homology group the second alternative of problem 6 holds. Use all this to calculate the first Betti number without any direct consideration of one-cycles.

ADDITIONAL PROBLEMS

A1. Prove that every n -dimensional simplicial polyhedron can be homeomorphically (in fact, piece-wise linearly) embedded into \mathbb{R}^{2n+1} .

A2. Calculate the fundamental group of the sphere with m handles and use this calculation to calculate the first homology group.

A3. Prove that every finitely presented group is isomorphic to the fundamental group of a finite two-dimensional simplicial polyhedron. *NOTE:* Do this problem only if you did not read the section of Rothman on Seifert-Van Kampen Theorem which contains the proof of Theorem 7.45.

A4. Let P be a compact convex polyhedron in \mathbb{R}^n , F_m , $m = 0, 1, \dots, n - 1$ be the numbers of its m -dimensional faces. Prove the *Generalized Euler Theorem*:

$$\sum_{m=0}^{m=n-1} F_m = 1 + (-1)^m.$$

Hint: You may use the fact that homology groups of simplicial polyhedron are independent of a simplicial decomposition.

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PROBLEM SET # 2

DEGREE, CELLULAR COMPLEXES AND CELLULAR HOMOLOGY

due on Tuesday 2-28-95

9. Every continuous map $f : S^n \rightarrow S^n$ such that $|\deg f| \neq 1$ has a fixed point.
10. Every continuous map $f : \mathbb{R}P(2n) \rightarrow \mathbb{R}P(2n)$ has a fixed point.
11. Give a detailed proof that for every $m \in \mathbb{Z}$ there exists a simplicial decomposition of the n -sphere S^n and a simplicial map $\phi : K \rightarrow K$ of the corresponding simplicial complex K of degree m .
Hint: Use induction and the construction of S^n as the “double cone” over S^{n-1} .
12. Prove that every finite one-dimensional CW complex allows a simplicial decomposition.
13. Let X be a CW complex which has a_k cells in dimension $2k$, $k = 0, 1, \dots$, Calculate homology groups of X .
14. Calculate homology groups of the Cartesian product of $S^m \times S^n$ using a cellular decomposition.
15. Let X be the set of all unit tangent vectors to the sphere S^2 with the natural topology induced from the embedding of S^2 into \mathbb{R}^3 (the unit tangent bundle). Prove that X allows a cellular decomposition and calculate its homology.
16. Find three linearly independent unit vector fields on S^3 . Use this fact to calculate the homology groups of the unit tangent bundle to S^3 .
17. Consider the following CW complex : its 1-skeleton is the circle S^1 with the standard cellular decomposition; there are m two-dimensional cells C_1, \dots, C_m and the identification of ∂C_k with the 1-skeleton is given by the rotation by $\frac{2\pi k}{m}$. Calculate homology groups of this complex.

ADDITIONAL PROBLEMS

A5. Let $p_n : S^n \rightarrow \mathbb{R}P(n)$ be the standard projection. Prove that for $n \geq 2$ no continuous map $f : \mathbb{R}P(n) \rightarrow \mathbb{R}P(1)$ can be lifted to a map $F : S^n \rightarrow S^1$ such that $p_1 \circ F = f \circ p_n$.

A6. Construct an example of a finite two-dimensional CW complex which does not allow a simplicial decomposition.

A7. Prove that the fundamental group of a cell polyhedron is the same as for its 2-skeleton. *Note:* Carefully justify any approximation you are going to use.

A8. Use simplicial approximation to prove *Hopf theorem*: Two maps of S^n into itself are homotopic if and only if they have the same degree.

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MID-TERM EXAMINATION

SIMPLICIAL AND CELL HOMOLOGY

Saturday 3-4-95

Do one problem from Section 1 and three problems from Section 2.

SECTION 1

1.1. Prove that if $m < n$ then any continuous map $f : S^m \rightarrow S^n$ is null-homotopic, ie homotopic to a map into a point.

1.2. Prove that there is no continuous map $f : S^n \rightarrow S^1$ for $n > 1$ which sends the opposite points into opposite points, ie $f \circ I_n = I_1 \circ f$ where I_k is the flip map on S^k .

1.3. Prove using the degree theory for the maps of the sphere that any non-constant polynomial with complex coefficients has a complex root.

SECTION 2

2.1. Let K be a finite n -dimensional simplicial complex with the following property: the union of interiors of its n - and $(n - 1)$ - dimensional simplexes is connected and every $(n - 1)$ - dimensional simplex belongs to exactly two n - dimensional simplexes. Let L be the complex obtained from K by eliminating one of n -dimensional simplexes. Suppose you know the Betti numbers of K .

Describe all possibilities for the Betti numbers of L .

2.2. Calculate homology groups of $S^2 \times \mathbb{R}P(2)$.

2.3. Consider the three-dimensional torus represented as the unit cube in \mathbb{R}^3 with pairs of opposite faces identified. Consider the group of order three generated by the rotation by $\frac{2\pi}{3}$ around one of the main diagonals. The factor-space possesses a natural cellular decomposition which is inherited from the standard decomposition of the torus.

Calculate the cellular homology of the factor-space.

2.4. Consider the space of oriented big circles in S^3 or, equivalently, the space of oriented two-dimensional subspaces of \mathbb{R}^4 with the natural topology.

Construct a cell decomposition of this space and calculate its homology groups.

2.5. Consider the following subset S of \mathbb{R}^3 :

$$S = \{(x, y, z) \in \mathbb{R}^3 : ((x^2 + y^2)^{\frac{1}{2}} - 1)^2 + z^2 = 1\}.$$

In other words, S is the surface of revolution around z axis of the circle in the xz plane with the center on the x axis which passes through the origin.

Construct a cellular decomposition of S and calculate its homology groups.

2.6. Consider the following two-dimensional cellular complex C . Its one skeleton C_1 is the circle identified with the unit circle in the complex plane (It can be viewed as a one-cell attached to a zero-cell). It has two two-dimensional cells c_1 and c_2 . The characteristic maps of c_1 and c_2 correspondingly have the form $\mathbb{D} \rightarrow C$ where \mathbb{D} is the unit disc in the complex plane. Their restrictions to the boundary $\partial\mathbb{D} \rightarrow C_1$ are the maps of the unit circle of the form $z \rightarrow z^4$ and $z \rightarrow z^5$ correspondingly.

Calculate homology groups of C .

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PROBLEM SET # 3

HIGHER HOMOTOPY GROUPS AND LOCALLY TRIVIAL FIBRATIONS

due on Thursday 03-16-95

18. Write down explicit formulas for the homotopies establishing commutativity of the homotopy groups $\pi_n(X, x_0)$ for $n \geq 2$ and the relative homotopy groups $\pi_n(X, A, x_0)$ for $n \geq 3$.

19. Prove that homotopically equivalent spaces have isomorphic homotopy groups. *Note:* Pay attention to the fact that the homotopy equivalences may not fix the base points.

20. Prove that all higher homotopy groups of the bouquet of $n \geq 1$ circles are trivial.

21. Calculate the higher homotopy groups of the Klein bottle.

22. Consider a fibration with the total space X , base B and fiber F . Suppose one of the three spaces is contractible and you know homotopy groups of one of the other two. Show how to find the homotopy groups of the remaining space.

23. Prove that $\pi_3(S^2)$ is an infinite group. *Hint: Use Hopf fibration.*

24. Prove that $\pi_k(\mathbb{C}P(n)) = 0$ for $3 \leq k \leq 2n$ and that $\pi_2(\mathbb{C}P(n))$ and $\pi_{2n+1}(\mathbb{C}P(n))$ are infinite groups.

25. Consider the unit tangent bundle of the sphere S^2 as locally trivial fibration with the base S^2 and the fiber S^1 . Prove that the map $\Delta : \pi_2(S^2) \rightarrow \pi_1(S^1)$ in the exact sequence on this fibration has non-trivial image.

26. A *Serre fibration* is a map $p : X \rightarrow B$ for which the lifting homotopy principle holds. Give an example of a Serre fibration where both X and B are compact connected metrizable spaces and which is not a locally trivial fibration.

ADDITIONAL PROBLEMS

A9. Prove that any locally trivial fibration whose base is a disc D^n is equivalent to the direct product.

A10. Calculate homotopy groups of the sphere with n handles.

A11. Give an example of a path-connected compact metric space all of whose homotopy groups are trivial and which is not contractible.

A12.(P.Foth) Prove that the tangent bundle to the direct product of spheres where at least one sphere has an odd dimension is equivalent to the direct product. In other words, if the dimension of our product space X is equal to n there are n linearly independent continuous vector fields on X .

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PROBLEM SET # 4

DIFFERENTIABLE MANIFOLDS

due on Tuesday 4-4-95

27. Describe the structure of differentiable manifold on the complex projective space $\mathbb{C}P(n)$ by explicitly defining coordinate charts and calculating transition functions.
28. Give a detailed description of the structure of differentiable manifold on the sphere with n handles represented as a regular $4n$ -gon with properly identified pairs of sides. Use the outline given in class on March 16.
29. Prove that *any* structure of differentiable manifold on the real line \mathbb{R} or on the circle S^1 is equivalent to the standard one.
30. Prove that the group $\text{Diff}(M)$ of diffeomorphisms of any connected differentiable manifold M acts transitively on M . *Hint:* First prove the required property locally.
31. Prove that the group $SO(3)$ of orthogonal 3×3 matrices with determinant one is an imbedded submanifold of the nine-dimensional Euclidean space of all 3×3 matrices.
32. Prove that $SO(3)$ with the differentiable structure described in the previous problem is diffeomorphic to the real projective space $\mathbb{R}P(3)$ with the standard differentiable structure. *Hint:* Represent an orthogonal transformation as a rotation around an axis.
33. Let M be a differentiable manifold and $f : M \rightarrow M$ a diffeomorphism. Consider the direct product $M \times [0, 1]$ with the identification of pairs of points $(0, f(x))$ and $(1, x)$ for all $x \in M$. Show that the resulting object which we denote M_f possesses a natural structure of differentiable manifold (suspension construction). Prove that M_f is a locally trivial fibration with base S^1 and the fiber M .
34. Apply suspension construction to the following three cases :
 (i) $M = \mathbb{R}$, $f(x) = -x$, (ii) $M = S^1$, $f(z) = -z$, (iii) $M = S^1$, $F(z) = \bar{z}$.
 Identify resulting manifolds. In which of these cases the fibration described in the previous problem turns out to be trivial?
35. Prove that any continuous real-valued function on a differentiable manifold can be arbitrarily well uniformly approximated by C^∞ functions.

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PROBLEM SET # 5

VECTOR FIELDS, DISTRIBUTIONS, DIFFERENTIAL FORMS

due on Tuesday 4-25-95

36. Prove that on any non-compact connected differentiable manifold there exists an incomplete smooth vector field. *Hint:* Use partition of unity.
37. Construct three linearly independent non-vanishing vector fields on S^3 and calculate their Lie brackets.
38. Consider the group H of 3×3 upper-diagonal matrices with the units on the diagonal (the Heisenberg group). This group has natural coordinates (x_{12}, x_{13}, x_{23}) and it acts on itself by left translations. Let v_{12}, v_{13}, v_{23} be the left-invariant vector-fields on H with the values at the identity $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ correspondingly. Consider the two-dimensional distributions E and F on H generated by v_{12}, v_{13} and v_{12}, v_{23} correspondingly. Calculate both distributions in the natural coordinates and show that E is integrable and F is not.
39. Suppose M is a compact differentiable manifold and $f : M \rightarrow \mathbb{R}$ is a C^2 function which has one non-degenerate minimum, one non-degenerate maximum and no more critical points. Prove that M is homeomorphic to S^n .
40. Prove that the tangent and cotangent bundle of any differentiable manifold are equivalent as vector bundles. Find a proper generalization of this statement to tensor bundles.
41. Prove that for the tangent bundle $T(M)$ one can always find another vector bundle E over M such that the Whitney sum of $T(M)$ and E is a trivial bundle *Hint:* Use an embedding theorem.
42. Prove that there is no non-vanishing skew-symmetric $2n$ differential form (a volume element) on the real projective space $\mathbb{R}P(2n)$, $n \geq 1$.
43. Construct volume elements (non-vanishing skew-symmetric differential forms of maximal dimension) on odd-dimensional real projective spaces and all complex projective spaces.
44. Use the definition of the Lie derivative for a tensor field Ω to calculate the Lie derivative of $\rho\Omega$ where ρ is a scalar differentiable function.

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PROBLEM SET # 6

You do not have to return written solutions

RIEMANNIAN METRICS, EXTERIOR DERIVATIVES,
ORIENTABILITY, DE RHAM COHOMOLOGY

45. Prove that the topology defined by the distance function generated by a Riemannian metric on a differentiable manifold coincides with the topology of the manifold.

46. Prove that the metric defined by any Riemannian metric on a compact differentiable manifold is complete. Prove that on any non-compact connected differentiable manifold M there exists a Riemannian metric which determines an incomplete metric on M .

47. Consider the standard embedding of the n -dimensional sphere S^n into R^{n+1} with the Riemannian metric induced by the embedding. Show that for any two points $x, y \in S^n$ which are not diametrically opposite there is a unique shortest curve in S^n connecting x and y , namely the shorter arc of the big circle. *Hint:* Use “geographical coordinates” on S^2 and induction in dimension.

48. Let ω be a non-vanishing differential 1-form. Prove that if $d\omega = \omega \wedge \alpha$ for some 1-form α then the codimension-one distribution $\text{Ker}\omega$ is integrable. *Hint:* Use Frobenius Theorem.

49. A volume element Ω on an n -dimensional manifold M determines a duality between differential $n - 1$ forms and vector fields on M via interior differentiation. Prove that closed forms correspond exactly to vector fields preserving Ω , ie divergence-free vector fields.

50. Prove that any *complex manifold* is orientable. A complex manifold is a differentiable manifold which has an atlas of coordinate neighborhoods modeled on C^n and such that transition maps are given by holomorphic functions of n variables.

51. For what values of m and n the space $\mathbb{R}P(m) \times \mathbb{R}P(n)$ is orientable?

52. Describe a basis in the first cohomology group of the sphere with n handles, ie construct $2n$ closed 1-forms $\omega_1, \dots, \omega_{2n}$ whose cohomology classes form a basis in the cohomology group. Describe the multiplicative structure in the cohomology ring. *Hint:* You may (but do not have to) use Problem 49.

ADDITIONAL PROBLEMS

A13. Suppose M and N are differentiable manifolds and $f : M \rightarrow N$ is a bijection. Prove that f is a diffeomorphism if and only if both f and its inverse carry differentiable functions into differentiable functions.

A14. Prove existence of a Hausdorff connected non-separable differentiable manifold.

ALTERNATIVE PROBLEMS FOR THE MID-TERM

(more difficult than 4.1)

X2. (maybe replace 1.1)

X3. Consider an n -dimensional cell complex X . Let $0 \leq k < n$ and let Y be the complex obtained by identifying the k -skeleton of X into a point. Describe the homology of Y in terms of the cell structure and homology of X . (maybe replace 2.2)

X5. For a continuous map $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ (or maybe $\mathbb{T}^n \rightarrow S^1$) show that if the induced map $f_* : H_1(\mathbb{T}^2) \rightarrow H_1(\mathbb{T}^2)$ is zero then f is null-homotopic.

3.2. Consider the following map of the n -dimensional torus to itself :

$$f(x_1, \dots, x_n) = (k_1x_1, \dots, k_nx_n) \pmod{1}$$

where k_1, \dots, k_n are integers. Find the induced map

$$(f_*)_n : H_n(\mathbb{T}^n) \rightarrow H_n(\mathbb{T}^n).$$

4.1. Show that for any natural number n there exists a compact manifold whose universal cover is S^3 and whose fundamental group is the cyclic group of order n .

4.2. A *line field* on a manifold M is a continuous map which assigns to each point $x \in M$ a one-dimensional subspace in the tangent space T_xM . A line field on $\mathbb{R}P(n)$ is, by definition, the projection of a flip-invariant line field on S^n . Prove that there is no line field on $\mathbb{R}P(2n)$.

TOPOLOGY/GEOMETRY PH.D QUALIFYING EXAMINATION

May 11, 1995

In order to obtain perfect score you should give complete solutions of two problems from each of the sections below. Partial credits will be given. If you use any result not discussed in the Math 527-528 series you should provide a proof of such a result.

Section 1: General topology and fundamental group

1.1. Prove that the complements to all dense countable subsets of \mathbb{R}^2 are homeomorphic.

1.2. A closed bounded subset X of \mathbb{R}^n is called *star-shaped* if for some point $x \in X$ every half-line which begins at x intersects the boundary of X at exactly one point. Prove that any closed bounded star-shaped set in \mathbb{R}^n is homeomorphic to the standard n -simplex σ^n .

1.3. Prove that the fundamental group of the set $[0, 1] \times [0, 1] \setminus K \times K$, where K is the standard Cantor set, is uncountable.

1.4. Consider the following two topologies in \mathbb{R}^2 : (i) *Zariski topology*, where a base of open sets is formed by the complements of the zero set of polynomials in two variables and (ii) the weakest topology where all straight lines are closed sets. Prove that the topological spaces thus obtained are not homeomorphic.

Section 2: Simplicial and cell homology, including connections with fundamental group.

2.1. Find $\pi_1(SO(n))$, $n \geq 4$.

2.2. Prove that for any finitely generated abelian group G there exists a compact connected manifold whose first homology group is isomorphic to G .

2.3. Consider the following set in \mathbb{R}^3 :

$$(((|x_1| - 3)^2 + x_3^2)^{\frac{1}{2}} - 2)^2 + x_2^2 = 1$$

Prove that it is a cell polyhedron and calculate its homology groups.

2.4. Let B_2 be the “figure eight” ie the bouquet of two circles. Calculate homology groups of $B_2 \times B_2$.

Section 3: Higher homotopy, fibered bundles, differentiable manifolds

3.1. Let Ω be a volume element on a differentiable manifold M . Prove that there exists a Riemannian metric on M such that its Riemannian volume coincides with (the absolute value of) Ω .

3.2. Consider the space of all straight lines in \mathbb{R}^3 with the natural structure of differentiable manifold. Prove that it is not orientable.

3.3. Let $f = (f_1, \dots, f_k) : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be a differentiable map and $M = f^{-1}(0)$. Assume that the rank of Df on M is equal to k at every point. Prove that M is an orientable $n - k$ -dimensional embedded submanifold of \mathbb{R}^n .

3.4. Consider the following differential 1-form in \mathbb{R}^3 : $\omega = x_1 dx_2 - x_2 dx_1 + dx_3$. Show that for any non-zero scalar function ρ the form $\rho\omega$ is not closed.

3.5. Let $f : S^n \rightarrow \mathbb{T}^n, n \geq 2$ be a differentiable map. Prove that it has *degree zero*, ie for any volume element ω on \mathbb{T}^n the n -form $f^*\omega$ has zero integral over the sphere S^n .